Lecture notes

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Abstract

These notes are a compilation of separate original notes by Jim Nolen and Pierre Cardaliaguet, with some very small extra background material, all errors are mine. This is not my text. Whenever "we' appears below, it should be understood as "Pierre Cardaliaguet".

1 What is a mean-field game?

The mean-field game system consists of a Hamilton-Jacobi equation for a value function $u(x,t)$ and a Fokker-Planck equation for a mean-field density $m(x,t)$:

$$\begin{array}{ll} -\partial_t u - \nu\Delta u + H(x, Du) = f(x, m(x,t)) & \text{in } \mathbb{R}^d \times (0,T) \\ \partial_t m - \nu\Delta m - \operatorname{div}(D_p H(x, Du) m) = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ m(0,x) = m_0(x)\ ,\ u(x,T) = G(x). & \end{array} \tag{1.1}$$

As we will see, this system comes up in an optimization problem approximating a large number of agents (players), where the behavior of each player is governed by the rest of the players via the mean-field. Accordingly, the evolution of the value function $u(x,t)$ for an individual player, is coupled to that of the density $m(x,t)$ of all players. The term mean-field refers to the fact that the strategy of each player is affected only by the average density (mean-field) of the other players, and not by a particular stochastic configuration of the system. The function $H(x,p)$ is the Hamiltonian, and the function $f(x,m)$ is a local coupling between the value function of the optimal control problem and the density of the players. Of course, the coupling need not be local, and we will consider non-local couplings as well.

The most unusual feature of (5.1) is that it couples the forward Fokker-Planck equation that has an initial condition for $m(0,x)$ at the initial time $t=0$ to the backward in time Hamilton-Jacobi equation for $u(t,x)$ that has a prescribed terminal value at $t=T$. Thus, this is not a Cauchy problem that normally arises in PDE problems, and has novel features compared to what we are used to see.

Mean field game theory is devoted to the analysis of differential games with infinitely many players. For such large population dynamic games, it is unrealistic for a player to collect detailed state information about all other players. Fortunately this impossible task is

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useless: mean field game theory explains that one just needs to implement strategies based on the distribution of the other players. Such a strong simplification is well documented in the (static) game community since the seminal works of Aumann [14]. However, for differential games, this idea has been considered only very recently: the starting point is a series of papers by Lasry and Lions [102, 103, 104, 105], who introduced the terminology in around 2005. The term mean field comes for an analogy with the mean field models in mathematical physics, which analyse the behavior of many identical particles (see for instance Sznitman's notes [122]). Here, the particles are replaced by agents or players, whence the name of mean field games. Related ideas have been developed independently, and at about the same time, by Caines, Huang and Malhamé [88, 89, 90, 91], under the name of Nash certainty equivalence principle.

The Cardaliaguet notes we are following (literally copy-pasting almost 100% of the time) aim to basic presentation of the topic. They are largely based on Lions' series of lectures at the College de France [108] and on Lasry and Lions seminal papers on the subject [102, 103, 104, 105], but also on other notes taken from Lions lectures: Yves Achdou's survey for a CIME course [4] and Guant's notes [82] (see also the survey by Gomes and Saude [72]).

There are several approaches to the analysis of differential games with an infinite number of agents. A first one is to look at the limit of Nash equilibria in differential games with a large number of players and try to pass to the limit as this number tends to infinity. A second approach consists in guessing the equations that Nash equilibria of differential games with infinitely many players should satisfy and to show that the resulting solutions of these equations allow to solve differential games with finitely many players.

Concerning the first approach, little was completely understood until very recently. Lions explains in [108] how to derive formally an equation for the limit to Nash equilibria: it is a nonlinear transport equation in the space of measures (the "master equation"). Existence, uniqueness of solution for this equation is an open problem in general, and, beside the linear-quadratic case, one did not know how to pass to the limit is the Nash system. Progress has been made very recently on both questions [29, 51, 62] and we explain some of the ideas in the second part of these notes. The starting point is that, as observed by Lions [108], the "characteristics" of the infinite dimensional transport equations solve—at least formally—a system coupling of a Hamilton-Jacobi equation with a Kolmogorov-Fokker-Planck equation: this is the MFG system, which is the main object of the first chapter of these notes.

A very nice point is that this system also provides a solution to the second approach: indeed, the feedback control, given by the solution of the mean field game system, provides $\varepsilon-$Nash equilibria in differential games with a large (but finite) number of players. This point was first noticed by Huang, Caines and Malham [89] and further developed in several papers (Carmona, Delarue [44], Kolokoltsov, Li, Yang [95], etc...).

To complete the discussion on the master equation, let us finally underline that, beside the MFG system, another possible and natural simplification of this equation is a space discretization, which yields to a more standard transport equation in finite space dimension: see the discussion by Lions in [108], by Gomes, Mohr, Souza [63, 64, 65] and Guant [80].

We now describe the field field game system in a more precise way. The system has two unknowns u and m, which solve the equations

$$\left\{\begin{array}{ll} (i) & -\partial_t u - \nu\Delta u + H(x,m,Du) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T) \\ (ii) & \partial_t m - \nu\Delta m - \operatorname{div}\left(D_p H(x,m,Du) m\right) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T) \\ (iii) & m(0) = m_0\ ,\ u(x,T) = G(x,m(T)) \end{array}\right. \tag{1.2}$$

In the above system, ν is a nonnegative parameter. The first equation has to be understood backward in time and the second one is forward in time. There are two crucial structure conditions for this system: the first one is the convexity of $H = H(x,m,p)$ with respect to the last variable. This condition means that the first equation (a Hamilton-Jacobi equation) is associated with an optimal control problem. This first equation is interpreted as the value function associated with a typical small player. The second structure condition is that m_0 (and therefore $m(t)$) is (the density of) a probability measure.

The heuristic interpretation of this system is the following. An average agent controls the stochastic differential equation

$$dX_t = \alpha_t dt + \sqrt{2\nu} dB_t$$

where (B_t) is a standard Brownian motion. He aims at minimizing the quantity

$$\mathbb{E}\left[\int_0^T \frac{1}{2} L(X_s, m(s), \alpha_s) ds + G(X_T, m(T))\right]\ ,$$

where L is the Legendre transform of H with respect to the p variable. Note that in this cost the evolution of the measure $m(s)$ enters as a parameter.

The value function of our average player is then given by (1.2-(i)). His optimal control is—at least heuristically—given in feedback form by $\alpha^*(x,t) = -D_p H(x,m,Du)$. Now, if all agents argue in this way and if their associated noises are independent, then by the law of large numbers their repartition moves with a velocity which is due, on the one hand, to the diffusion, and, one the other hand, on the drift term $-D_p H(x,m,Du)$. This leads to the Kolmogorov-Fokker-Planck equation (1.2-(ii)).

The aim of these notes is to collect—with detailed proofs—various existence and uniqueness results obtained by Lasry and Lions for the above system when the Hamiltonian H is "separated": $H(x,m,p) = H(x,p) - F(x,m)$, F being a coupling between the two equations. There are two types of coupling which are appear in the mean field game literature: either F is nonlocal and regularizing, i.e., we see $F = F(x,m(t))$ as a map on the space of probability measures. This is typically the case when two players who are not to close from each other can influence themselves. Or F is of local nature, i.e., $F = F(x,m(x,t))$ depends on the value of the density at the point (t,x), meaning that the players only take into account their very nearest neighbors. Although the former coupling can be seen as a limit case of the first one, in practice techniques of proof are more depending in this case. In particular, if we provide existence and uniqueness results for nonlocal couplings when $\nu = 1$ (viscous case) and $\nu = 0$ (1rst order model), we consider local couplings only for viscous equations (i.e., $\nu = 1$). We carefully avoid the case of 1rst order models ($\nu = 0$) with local coupling: this case, described in [108], is only understood under specific structure conditions and requires several a priori estimates which, unfortunately, exceed the modest framework of these notes.

Some comments on the literature are now in order. Since the pioneering works by Lasry and Lions and by Huang, Caines and Malham, the literature on the MFG has grown very fast: it is by now almost impossible to give a reasonable account of the activity on the topic. Many references on the subject can be found, for instance, in the survey by Gomes and Saud [72] and in the monograph by Bensoussan, Frehse and Yam [25]. We only provide here a few references, without the smallest pretension of exhaustivity.

By nature mean field games are related with probability and partial differential equations. Both community have a different approach of the topic, mostly inspired by the works of Caines, Huang and Malham for the probability part, and by Lasry and Lions for the PDE one.

Let us start with the probabilistic aspects. As the value function of an optimal control problem is naturally described in terms of backward stochastic differential equations (BSDEs), it is very natural to understand the MFG system as a BSDE with a mean field term of McKean-Vlasov type: this is the approach generally followed the probabilistic part of the literature on mean field games: beside the papers by Huang, Caines and Malham already quoted, see also Buckdahn, Li, Peng [28], Buckdahn, Djehiche, Li, Peng [28], Andersson, Djehiche [13] (where a linear MFG system appears as optimality condition of a control of mean field type). Forward-backward stochastic differential equation (FBSDE) of the McKean-Vlasov type, are analyzed by Carmona, Delarue [44], Kolokoltsov, Li, Yang [95] (with nonlinear diffusions). MFG models with a major player are discussed by Huang [85], while Nourian, Caines, Malhame, Huang [115] deal with mean field LQG control in leader-follower stochastic multi-agent systems. Differential games in which several major players influence an overall population but compete with each others lead to differential games of mean field type, as considered by Bensoussan, Frehse [24]. Linear quadratic MFG system have also been very much investigated: beside Huang, Caines and Malham work, see Bensoussan, Sung, Yam, Yung [23], Carmona, Delarue [43] for probabilistic arguments, and Bardi [15] from a PDE view point.

In terms of PDE, the analysis of mean field games boils down—more or less—to solve the coupled system (1.2) with various assumptions on the coefficients. Beside Lasry and Lions' papers, other existence results and estimates for classical MFG system can be found in Guant [77, 81] (by use of Hopf-Cole transform for 2nd order of MFG systems with local coupling), Cardaliaguet, Lasry, Lions, Porretta [40] (2nd order MFG systems with local unbounded coupling), Bardi, Feleqi [16] (stationary MFG systems with general diffusions and boundary conditions), Gomes, Pirez, Sanchez-Morgado [66] (estimates for stationary MFG systems), Cardaliaguet [38] (1rst order MFG system, local coupling by methods of calculus of variation). Models with several populations are discussed by Feleqi [56], Bardi, Feleqi [16], Cirant [47]. Other models are considered in the literature: the so-called extended mean field games, i.e., MFG systems in which the HJ equation also depends on the velocity field of the players have been studied by Gomes, Patrizi, Voskanyan [67], Gomes, Voskanyan [68]; Santambrogio [119] discusses MFG models with density constraints; mean field kinetic model for systems of rational agents interacting in a game theoretical framework is discussed in [49] and [50].

Numerical aspects of the theory have been developed in particular by Achdou, Capuzzo Dolcetta [1], Achdou, Camilli, Capuzzo Dolcetta [2], [3], Achdou, Perez [6] Camilli, Silva [32], Lachapelle, Salomon, Turinici [97].

As shown by numerical studies, solutions of time dependent MFG systems, such as (1.2) quickly stabilize to stationary MFG systems: the analysis of the phenomenon (i.e., the long time behavior of solutions of the mean field game system) has been considered for discrete systems by Gomes, Mohr, Souza [63] and for continuous ones in Lions' lectures, and subsequently developed by Cardaliaguet, Lasry, Lions, Porretta [40, 41] for second order MFG game system with local and nonlocal couplings, in Cardaliaguet [37], from 1rst order MFG systems with nonlocal coupling.

It is impossible to cover all the applications of MFG to economics, social science, biological science, and engineering—and this part is even less complete than the previous ones. Let us just mention that the early work on large population differential games was motivated by wireless power control problems: see Huang, Caines, Malham [86, 87]. Application to economic models can be found in Guant [76], Guant, Lions, Lasry, [78, 106], Lachapelle [96], Lachapelle, Wolfram [98], Lucas, Moll [109]. A price formation model, inspired by the MFG, has been introduced in Lasry, Lions [102] and analyzed by Markowich, Matevosyan, Pietschmann, Wolfram [110], Caffarelli, Markowich, Wolfram [31].

2 Hamilton-Jacobi equations

This section is an edited and expanded version of Jim Nolen's notes.

We first recall some basic facts about the solutions of the initial (or, rather, terminal) value problem for the Hamilton-Jacobi equations of the form

$$\begin{aligned} &u_t + H(\nabla u, x) = 0 \\ &u(T,x) = u_0(x). \end{aligned} \tag{2.1}$$

In order to explain how such problems come about, and to understand why the mean-field games with their coupling of forward and backward equations are so natural, we need to recall some basic notions from the control theory.

2.1 Deterministic optimal control

Consider the following abstract optimization problem. Let $y(s) : [t,T] \to \mathbb{R}^d$ denote the state of a system at a time $s \in [t,T]$, which evolves according to a system of ordinary differential equations

$$\begin{aligned} &\dot{y}(s) = f(y(s), \alpha(s)), \quad s \in [t,T] \\ &y(t) = x \in \mathbb{R}^d, \end{aligned} \tag{2.2}$$

with $t > 0$ and $x \in \mathbb{R}^d$ fixed. The function $\alpha(s)$ is called a control and takes values in a compact subset A of $\mathbb{R}^m$. The set of all admissible controls will be denoted by $\mathcal{A}_{t,T}$:

$$\mathcal{A}_{t,T} = \{\alpha(s) : [t,T] \to A \mid \ \alpha(s) \text{ is measureable}\}\,. \tag{2.3}$$

When the dependence on t and T is clear from the context, we will simply use $\mathcal{A}$ instead. By choosing α we have control over the course of the system $y(t)$.

We would like to control the system in an optimal way, in the following sense. Let a function $g(x) : \mathbb{R}^d \to \mathbb{R}$ represent a final payoff, which depends on the final state of the system at time T, and the function $r(x, \alpha) : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ represent a running payoff or running cost. Given the initial state of the system $y(t) = x$, the optimization problem is to find an optimal control $\alpha^*(\cdot)$ that maximizes the functional

$$J_{x,t}(\alpha^*) = \max_{\alpha(\cdot)\in\mathcal{A}} J_{x,t}(\alpha) = \max_{\alpha(\cdot)\in\mathcal{A}} \Big[\int_t^T r(y(s), \alpha(s))\, ds + g(y(T)) \Big]. \tag{2.4}$$

Even if an optimal control does not exist, we may study the function

$$u(x,t) = \sup_{\alpha(\cdot)\in\mathcal{A}} J_{x,t}(\alpha) = \max_{\alpha(\cdot)\in\mathcal{A}} \Big[\int_t^T r(y(s), \alpha(s))\, ds + g(y(T)) \Big], \tag{2.5}$$

called the value function associated with the control problem. It depends on x and t through the initial conditions defining $y(s)$. Note that $u(x,T) = g(x)$, hence whatever evolution problem in t the function $u(x,t)$ satisfies, it is natural to prescribe its terminal value at $t = T$ and not its initial value at $t = 0$.

2.2 The Dynamic Programming Principle

Theorem 2.1. *Let $u(x,t)$ be the value function defined by (2.5). If $t < \tau \leq T$, then*

$$u(x,t) = \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^\tau r(y(s), \alpha(s))\, ds + u(y(\tau), \tau) \right]. \tag{2.6}$$

The relation (2.32), called the Dynamic Programming Principle, is a fundamental tool in the analysis of optimal control problems. It says that if we know the value function at time $\tau > t$, we may determine the value function at time t by optimizing from time t to time τ and using $u(\cdot, \tau)$ as the payoff. Roughly speaking, this is reminiscent of the Markov property of a stochastic process, in the sense that if we know $u(x,\tau)$ we can determine $u(\cdot, t)$ for $t < \tau$ without any other information about the control problem beyond time τ, for times $s \in [\tau, T]$. More precisely, it means that $u(x,t)$ satisfies a semi-group property. Note, however, that the time in the semi-group property is running backwards!

Proof of Theorem 2.1: The proof of the Dynamic Programming Principle is based on the simple observation that any admissible control $\alpha \in \mathcal{A}_{t,T}$ is at combination of a control in $\mathcal{A}_{t,\tau}$ with a control in $\mathcal{A}_{\tau,T}$. We will express this relationship as

$$\mathcal{A}_{t,T} = \mathcal{A}_{t,\tau} \oplus \mathcal{A}_{\tau,T}. \tag{2.7}$$

This notation $\oplus$ means that if $\alpha_t(s) \in \mathcal{A}_{t,\tau}$ and $\alpha_\tau(s) \in \mathcal{A}_{\tau,T}$, then the control defined by splicing α_t and α_τ according to

$$\alpha(s) = (\alpha_t \oplus \alpha_\tau)(s) := \left\{ \begin{array}{ll} \alpha_t(s), & s \in [t,\tau] \\ \alpha_\tau(s), & s \in [\tau, T] \end{array} \right. \tag{2.8}$$

is an admissible control in $\mathcal{A}_{t,T}$. On the other hand, if we have $\alpha \in \mathcal{A}_{t,T}$, then by restricting the domain of α to $[t,\tau]$ we obtain an admissible control in $\mathcal{A}_{t,\tau}$. Similarly, by restricting the domain of α to $[\tau,T]$ we obtain an admissible control in $\mathcal{A}_{\tau,T}$.

The function u is defined as

$$\begin{aligned} u(x,t) &= \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^T r(y(s),\alpha(s))\,ds + g(y(T))\right] \\ &= \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^\tau r(y(s),\alpha(s))\,ds + \int_\tau^T r(y(s),\alpha(s))\,ds + g(y(T))\right]. \end{aligned}$$

Notice that the first integral on the right depends only on y and α up to time τ, while the last two terms depend on the values of y and α after time τ. Since a control $\alpha \in \mathcal{A}_{t,T}$ may be decomposed as $\alpha = \alpha_1 \oplus \alpha_2$ with $\alpha_1 \in \mathcal{A}_{t,\tau}$ and $\alpha_2 \in \mathcal{A}_{\tau,T}$, we may maximize over each component in the decomposition:

$$\begin{aligned} u(x,t) &= \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^\tau r(y(s),\alpha(s))\,ds + \int_\tau^T r(y(s),\alpha(s))\,ds + g(y(T))\right] \\ &= \max_{\alpha_1\in\mathcal{A}_{t,\tau},\alpha_2\in\mathcal{A}_{\tau,T},\alpha=\alpha_1\oplus\alpha_2} \left[\int_t^\tau r(y(s),\alpha(s))\,ds + \int_\tau^T r(y(s),\alpha(s))\,ds + g(y(T))\right]. \end{aligned}$$

On the right side, the system state $y(t)$ is determined by (2.2) with $\alpha = \alpha_1 \oplus \alpha_2 \in \mathcal{A}_{t,T}$. Therefore, we may decompose the system state as $y(s) = y_1 \oplus y_2$ where $y_1(s) : [t,\tau] \to \mathbb{R}^d$ and $y_2(s) : [\tau,T] \to \mathbb{R}^d$ are defined by

$$\begin{aligned} y_1'(s) &= f(y_1(s),\alpha_1(s)), \quad s \in [t,\tau] \\ y_1(t) &= x \end{aligned}$$

and

$$\begin{aligned} y_2'(s) &= f(y_2(s),\alpha_2(s)), \quad s \in [\tau,T] \\ y_2(\tau) &= y_1(\tau) = y(\tau). \end{aligned}$$

Here we use $\oplus$ to denote the splicing or gluing of y_1 and y_2 to create $y(t) : [t,T] \to \mathbb{R}^d$. Therefore, we have

$$u(x,t) = \max_{\alpha_1\in\mathcal{A}_{t,\tau}} \max_{\alpha_2\in\mathcal{A}_{\tau,T},y_2(\tau)=y_1(\tau)} \left[\int_t^\tau r(y_1(s),\alpha_1(s))\,ds + \int_\tau^T r(y_2(s),\alpha_2(s))\,ds + g(y_2(T))\right],$$

where the initial point for $y_2(\tau)$ is $y_2(\tau) = y_1(\tau)$. Observe that y_1 depends only on x and α_1, not on y_2 or α_2. Since the first integral depends only on α_1 and y_1, this may be rearranged as

$$\begin{aligned} u(x,t) &= \max_{\alpha_1\in\mathcal{A}_{t,\tau}} \max_{\alpha_2\in\mathcal{A}_{\tau,T},y_2(\tau)=y_1(\tau)} \left[\int_t^\tau r(y_1(s),\alpha_1(s))\,ds + \int_\tau^T r(y_2(s),\alpha_2(s))\,ds + g(y_2(T))\right] \\ &= \max_{\alpha_1\in\mathcal{A}_{t,\tau}} \left[\int_t^\tau r(y_1(s),\alpha_1(s))\,ds + \max_{\alpha_2\in\mathcal{A}_{\tau,T},y_2(\tau)=y_1(\tau)} \left(\int_\tau^T r(y_2(s),\alpha_2(s))\,ds + g(y_2(T))\right)\right] \\ &= \max_{\alpha_1\in\mathcal{A}_{t,\tau}} \left[\int_t^\tau r(y_1(s),\alpha_1(s))\,ds + u(y_1(\tau),\tau)\right] \qquad \text{(using the definition of } u\text{)} \\ &= \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^\tau r(y(s),\alpha(s))\,ds + u(y(\tau),\tau)\right] \qquad (2.9) \end{aligned}$$

This completes the proof. $\square$

Notice that in this proof we have not assumed that an optimal control exists.

2.3 The Hamilton-Jacobi-Bellman Equation

How does the value function depend on x and t? Is it continuous in (x,t)? Is it differentiable? Does it satisfy a PDE? Unfortunately, the value function may be not differentiable, as shown by the following simple example. Suppose that $f(x,\alpha) = \alpha$, $g \equiv 0$, and $r(x,\alpha)$ is defined by

$$r(x,\alpha) = -\mathbb{I}_D(x) = \begin{cases} -1, & x \in D \\ 0, & x \in \mathbb{R}^d \setminus D \end{cases} \tag{2.10}$$

where $D \subset \mathbb{R}^d$ is a bounded set. Suppose that the set of admissible controls is $A = \{|\alpha| \le 1\}$, and $y'(s) = \alpha(s)$, so that $|y'(s)| \le 1$. Therefore, the value function may be written as

$$u(x,t) = \max_{y:[t,T]\to\mathbb{R}^d, |y'|\le 1,\, y(t)=x} \left[\int_t^T -\mathbb{I}_D(y(s))\,ds \right]. \tag{2.11}$$

Clearly $u(x,t) \le 0$, and the optimum is obtained by paths that spend the least amount of time in the set D. If $x \in \mathbb{R}^d \setminus D$, then $u(x,t) = 0$, because we could take $y(s) = x$ for all $s \in [t,T]$. In this case, the system state doesn't change, so the integral is zero, which is clearly optimal. On the other hand, if $x \in D$ then the optimal control moves $y(s)$ to $\mathbb{R}^d \setminus D$ as quickly as possible and then stays outside D. Since $|y'(s)| \le 1$, this implies that the value function is given explicitly by

$$u(x,t) = -\min\left((T-t), \operatorname{dist}(x, \mathbb{R}\setminus D)\right) \tag{2.12}$$

where

$$\operatorname{dist}(x, \mathbb{R}\setminus D) = \inf_{y\in\mathbb{R}\setminus D} |x-y|, \tag{2.13}$$

is the Euclidean distance from x to the outside of D. Albeit continuous, this function may be not differentiable. Indeed, if $D = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 1\}$ is the unit disk, then

$$u(x,t) = \begin{cases} |x| - 1, & |x| \le 1 \\ 0, & |x| \ge 1 \end{cases} \tag{2.14}$$

for $t \le T-1$. Thus $u(x,t)$ is not differentiable at the origin $x = (x_1,x_2) = (0,0)$ for $t < T-1$.

So, in general, the value function may be not differentiable. However, one can still derive a PDE satisfied by the value function. If the value function is differentiable, this equation is satisfied in the classical sense. At points where the value function is not differentiable, one can show that the value function (assuming it is at least continuous) satisfies the PDE in a weaker sense. This weaker notion of "solution" is called a "viscosity solution" of the PDE. For the moment, we will formally compute as if the value function were actually differentiable.

Let us use the Dynamic Programming Principle to formally derive an equation solved by the value function $u(x,t)$. The Dynamic Programming Principle does not require differentiability of the value function; however, in our computations we assume that the value function is continuous and differentiable with respect to both x and t, and that the optimal control $\alpha^*(t)$ is continuous in time. The Dynamic Programming Principle tells us that

$$u(x,t) = \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^\tau r(y(s),\alpha(s))\,ds + u(y(\tau),\tau) \right]. \tag{2.15}$$

To formally derive a PDE for u, we let $h \in (0, T-t)$ and set $\tau = t+h < T$, then

$$u(x,t) = \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^{t+h} r(y(s),\alpha(s))\,ds + u(y(t+h),t+h) \right]. \tag{2.16}$$

We'll assume that nearly optimal controls are approximately constant for $s \in [t, t+h]$.

First, consider the term $u(y(t+h), t+h)$. From the chain rule and our assumption that u is continuously differentiable in x and t, we conclude that

$$\begin{aligned} u(y(t+h),t+h) &= u(y(t),t) + hy'(t)\cdot\nabla u(y(t),t) + hu_t(y(t),t) + o(h) \quad (2.17)\\ &= u(y(t),t) + hf(y(t),\alpha(t))\cdot\nabla u(y(t),t) + hu_t(y(t),t) + o(h)\\ &= u(x,t) + hf(x,\alpha(t))\cdot\nabla u(x,t) + hu_t(x,t) + o(h). \quad (2.18) \end{aligned}$$

Now, plug this into (2.16):

$$u(x,t) = \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^{t+h} r(y(s),\alpha(s))\,ds + u(x,t) + hf(x,\alpha(t))\cdot\nabla u(x,t) + hu_t(x,t) + o(h) \right]. \tag{2.19}$$

The term $u(x,t)$ may be pulled out of the maximum, so that it cancels with the left side:

$$0 = hu_t(x,t) + o(h) + \max_{\alpha(\cdot)\in\mathcal{A}} \left[\int_t^{t+h} r(y(s),\alpha(s))\,ds + hf(x,\alpha(t))\cdot\nabla u(x,t) \right]. \tag{2.20}$$

Now, divide by h and let $h \to 0$.

$$0 = u_t(x,t) + \frac{o(h)}{h} + \max_{\alpha(\cdot)\in\mathcal{A}} \left[\frac{1}{h}\int_t^{t+h} r(y(s),\alpha(s))\,ds + f(x,\alpha(t))\cdot\nabla u(x,t) \right]. \tag{2.21}$$

If $\alpha(s)$ is continuous at t, then as $h \to 0$,

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} r(y(s),\alpha(s))\,ds = r(y(t),\alpha(t)) = r(x,\alpha(t)) \tag{2.22}$$

So, if the nearly optimal controls are continuous for $s \in [t, t+h]$, then by letting $h \to 0$ in (2.21) we conclude that

$$u_t(x,t) + \max_{a\in A}\left[r(x,a) + f(x,a)\cdot\nabla u(x,t)\right] = 0, \quad x \in \mathbb{R}^d,\ t < T. \tag{2.23}$$

This equation is called the Hamilton-Jacobi-Bellman equation. The function $u(x,t)$ also satisfies the terminal condition

$$u(x,T) = g(x). \tag{2.24}$$

Notice that the HJB equation is a first-order, fully nonlinear equation, having the form

$$u_t + H(\nabla u, x) = 0 \tag{2.25}$$

where the Hamiltonian H is defined by

$$H(p, x) = \max_{a \in A} \left[r(x, a) + f(x, a) \cdot p \right], \quad p \in \mathbb{R}^d. \tag{2.26}$$

In addition to telling us how the value function depends on x and t, this PDE suggests what the optimal control should be. Suppose $u(x, t)$ is differentiable and solves (2.24)-(2.26) in the classical sense. Then the optimal control and the corresponding optimal trajectory are computed by finding $y^*(s)$ and $\alpha^*(s)$ which satisfy

$$\alpha^*(s) = \text{argmax}_{a \in A} \left[r(y^*(s), a) + f(y^*(s), a) \cdot \nabla u(y^*(s), s) \right]$$

and

$$\begin{aligned} \frac{dy^*(s)}{dt} &= f(y^*(s), \alpha^*(s)), \quad s > t \\ y^*(t) &= x. \end{aligned} \tag{2.27}$$

Infinite Time Horizon

So far, we have considered a deterministic control problem with a finite time horizon. This means that the optimization involves a finite time interval and may involve a terminal payoff. One might also consider an optimization problem posed on an infinite time interval. Suppose that $y : [t, \infty) \to \mathbb{R}^d$ satisfies

$$\begin{aligned} \dot{y}(s) &= f(y(s), \alpha(s)), \quad s \in [t, \infty) \\ y(t) &= x \in \mathbb{R}^d. \end{aligned} \tag{2.28}$$

Now the domain for the control is also $[t, \infty)$. We'll use $\mathcal{A} = \mathcal{A}_{t,\infty}$ for the set of admissible controls. For $x \in \mathbb{R}^d$, define the value function

$$u(x, t) = \max_{\alpha(\cdot) \in \mathcal{A}} J_{x,t}(\alpha) = \max_{\alpha(\cdot) \in \mathcal{A}} \left[\int_t^\infty e^{-\lambda s} r(y(s), \alpha(s))\, ds \right]. \tag{2.29}$$

The exponential term in the integral is a discount factor; without it, the integral might be infinite. Notice that there is no terminal payoff, only running payoff, and that the value function depends on t in a trivial way:

$$u(x, t) = e^{-\lambda t} u(x, 0). \tag{2.30}$$

So, to find $u(x, t)$ it suffices to compute

$$u(x) = \max_{\alpha(\cdot) \in \mathcal{A}} J_x(\alpha) = \max_{\alpha(\cdot) \in \mathcal{A}} \left[\int_0^\infty e^{-\lambda s} r(y(s), \alpha(s))\, ds \right] \tag{2.31}$$

where $\mathcal{A} = \mathcal{A}_{0,\infty}$.

Theorem 2.2 (Dynamic Programming Principle)**.** *Let $u(x)$ be the value function defined by (2.31). For any $x \in \mathbb{R}^d$ and $h > 0$,*

$$u(x) = \max_{\alpha(\cdot)\in\mathcal{A}_{0,h}} \left[\int_0^h e^{-\lambda s} r(y(s), \alpha(s))\, ds + e^{-\lambda h} u(y(h)) \right] \tag{2.32}$$

Proof: Exercise.

Using the Dynamic Programming Principle, one can formally derive the HJB equation for the infinite horizon control problem. The equation is:

$$-\lambda u + \max_{a\in A} \left[r(x,a) + f(x,\alpha) \cdot \nabla u \right] = 0 \tag{2.33}$$

which has the form

$$-\lambda u + H(\nabla u, x) = 0 \tag{2.34}$$

with the Hamiltonian $H(p, x)$ defined by

$$H(p,x) = \max_{a\in A} \left[r(x,a) + f(x,a) \cdot p \right] \tag{2.35}$$

Exercise: Check these computations.

2.4 Brief introduction to stochastic optimal control

Thus far, we have considered deterministic optimal control in which the dynamic behaviour of the system state is deterministic. In a stochastic optimal control problem, the state $y(s)$ is a stochastic process. Consequently, the controls also will be stochastic, since we may want to steer the system in a manner that depends on the system's stochastic trajectory. We now suppose that the system state $Y_s(\omega) : [t,T] \times \Omega \to \mathbb{R}^d$ satisfies a stochastic differential equation

$$\begin{aligned} dY_s &= f(Y_s, \alpha_s, s)ds + \sigma(Y_t, \alpha_s, s)dB_s, \quad s \geq t \\ Y_t &= x, \quad a.s. \end{aligned} \tag{2.36}$$

where B_s is a n-dimensional Brownian motion defined on probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\geq t}, P)$, and σ is a $d \times n$ matrix.

The control process $\alpha_s(\omega) : [t,T] \times \Omega \to \mathbb{R}^m$ is adapted to the filtration $\{\mathcal{F}_s\}_{s\geq t}$. The set of admissible controls is now

$$\mathcal{A}_{t,T} = \left\{ \alpha_s(\omega) : [t,T] \times \Omega \to A \mid \ \alpha_s \ \text{ is adapted to the filtration } \ \{\mathcal{F}_s\}_{s\geq t} \right\}. \tag{2.37}$$

The assumption that the controls are adapted means that we cannot look into the future; the control can only be chosen on the basis of information known up to the present time. Supposing that σ and f satisfy the usual bounds and continuity conditions, the stochastic process $Y_s(\omega)$ is uniquely determined by the initial condition $Y_t = x$ and the control process $\alpha_s(\omega)$.

Given a time $T > t$, the stochastic optimal control problem is to maximize

$$\max_{\alpha\in\mathcal{A}_{t,T}} J_{x,t}(\alpha(\cdot)) = \max_{\alpha\in\mathcal{A}_{t,T}} \mathbb{E}\left[\int_t^T r(Y_s, \alpha_s, s)\, ds + g(Y_T) \mid Y_t = x \right] \tag{2.38}$$

As before, the function $r(y, \alpha, s)$ represents a running payoff (or running cost, if $r < 0$), and g represents a terminal payoff (or terminal cost, if $g < 0$). Since the system state is a stochastic process, the net payoff is a random variable, and our goal is to maximize the expected payoff. Again, even if an optimal control process does not exist, we may define the value function as

$$u(x,t) = \max_{\alpha \in \mathcal{A}_{t,T}} \mathbb{E}\left[\int_t^T r(Y_s, \alpha_s, s)\, ds + g(Y_T) |\, Y_t = x\right] \tag{2.39}$$

Notice that the value function is not random.

In (2.39) the time horizon is finite. One could also pose an optimal control problem on an infinite time horizon. For example, one might consider maximizing

$$\max_{\alpha \in \mathcal{A}} J_{x,t}(\alpha(\cdot)) = \max_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^\gamma e^{-\lambda s} r(Y_s, \alpha_s)\, ds + e^{-\lambda \gamma} h(Y_\gamma)\right] \tag{2.40}$$

where γ is a stopping time.

2.5 Dynamic Programming Principle for Stochastic Control

For the stochastic control problem there is a Dynamic Programming Principle that is analogous to the DPP for deterministic control. Using the Markov Property of the stochastic process Y_t, one can easily prove the following:

Theorem 2.3. *Let $u(x,t)$ be the value function defined by (2.39). If $t < \tau \le T$, then*

$$u(x,t) = \max_{\alpha \in \mathcal{A}_{t,\tau}} \mathbb{E}\left[\int_t^\tau r(Y_s, \alpha_s, s)\, ds + u(Y_\tau, \tau) \mid Y_t = x\right] \tag{2.41}$$

Proof: Exercise. The idea is the same as in the case of deterministic control. Split the integral into two pieces, one over $[t, \tau]$ and the other over $[\tau, T]$. Then condition on $\mathcal{F}_\tau$ and use the Markov property, so that the second integral and the payoff may be expressed in terms of $u(Y_\tau, \tau)$. $\square$

2.6 The Hamilton-Jacobi-Bellman equation

Using the Dynamic Programming Principle, one can formally derive a PDE for the value function $u(x,t)$. As in the case of deterministic optimal control, one must assume that the value function is sufficiently smooth. Because the dynamics are stochastic, we want to apply Itô's formula in the way that we used the chain rule to derive the HJB equation for deterministic control. Thus, this formal computation requires that the value function by twice differentiable.

From Itô's formula we see that

$$\begin{aligned} u(Y_\tau, \tau) - u(x,t) &= = \int_t^\tau \left[u_t(Y_s, s) + f(Y_s, \alpha_s, s) \cdot \nabla u(Y_s, s)\right] ds \\ &+ \int_t^\tau \frac{1}{2} \sum_k \sum_{i,j} u_{x_i x_j}(Y_s, s) \sigma^{jk}(Y_s, \alpha_s, s) \sigma^{ik}(Y_s, \alpha_s, s)\, ds \end{aligned}$$

$$+\int_t^\tau (\nabla u(Y_s,s))^T \sigma(Y_s,\alpha_s,s)\,dB_s \tag{2.42}$$
$$= \int_t^\tau u_t(Y_s,s) + \mathcal{L}^\alpha u(Y_s,s)\,ds + \int_t^\tau (\nabla u(Y_s,s))^T \sigma(Y_s,\alpha_s,s)\,dB_s$$

where $\mathcal{L}$ is the second order differential operator

$$\begin{aligned}\mathcal{L}^\alpha u &= f(y,\alpha,s)\cdot\nabla u(y,s) + \frac{1}{2}\sum_k\sum_{i,j} u_{y_iy_j}(y,s)\sigma^{jk}(y,\alpha_s,s)\sigma^{ik}(y,\alpha_s,s)\\ &= f(y,\alpha,s)\cdot\nabla u(y,s) + \frac{1}{2}\mathrm{tr}(D^2u(y,s)\sigma(y,\alpha,s)\sigma^T(y,\alpha,s)),\end{aligned} \tag{2.43}$$

and D^2u is the matrix of second partial derivatives. Now we plug this into the DPP relation (2.41) and use the fact the martingale term in (2.43) has zero mean. We obtain:

$$\begin{aligned}0 &= \max_{\alpha\in\mathcal{A}_{t,\tau}} \mathbb{E}\left[\int_t^\tau r(Y_s,\alpha_s,s)\,ds + u(Y_\tau,\tau) - u(x,t) \mid Y_t = x\right]\\ &= \max_{\alpha\in\mathcal{A}_{t,\tau}} \mathbb{E}\left[\int_t^\tau r(Y_s,\alpha_s,s)\,ds + \int_t^\tau u_t(Y_s,s) + \mathcal{L}^\alpha u(Y_s,s)\,ds \mid Y_t = x\right].\end{aligned} \tag{2.44}$$

Finally, let $\tau = t+h$, divide by h and let $h\to 0$, as in the deterministic case. We formally obtain the HJB equation

$$u_t(x,t) + \max_{a\in A}\left[r(x,a,t) + \mathcal{L}^a u(x,t)\right] = 0. \tag{2.45}$$

This may be written as

$$u_t(x,t) + \max_{a\in A}\left[r(x,a,t) + \frac{1}{2}\mathrm{tr}(D^2u(x,t)\sigma(x,a,t)\sigma^T(x,a,t)) + f(x,a,t)\cdot\nabla u(x,t)\right] = 0 \tag{2.46}$$

which is, in general, a fully-nonlinear, *second order* equation of the form

$$u_t + H(D^2u, Du, x, t) = 0 \tag{2.47}$$

Notice that the equation is deterministic. The set of possible control values $A\subset\mathbb{R}^m$ is a subset of Euclidean space, and the maximum in the HJB equation (2.45) is over this deterministic set, not over the set $\mathcal{A}$.

HJB for the infinite horizon problem

Deriving the HJB for the infinite horizon problem is very similar. Let the value function be

$$u(x) = \max_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-\lambda s} r(Y_s,\alpha_s)\,ds \mid Y_0 = x\right], \tag{2.48}$$

and $\sigma(x,a)$, $f(y,a)$ and $r(y,a)$ be independent of t. Then the Dynamic Programming Principle shows that for any $\tau > 0$

$$u(x) = \max_{\alpha\in\mathcal{A}} E\left[\int_0^\tau e^{-\lambda s} r(Y_s,\alpha_s)\,ds + e^{-\lambda\tau}u(Y_\tau) \mid Y_0 = x\right]. \tag{2.49}$$

Using Itô's formula as before, we formally derive the second order equation equation

$$-\lambda u(x) + \max_{a\in A}\left[r(x,a) + \mathcal{L}^a u(x)\right] = 0 \tag{2.50}$$

2.7 The basic theory of Hamilton-Jacobi equations

The Euler-Lagrange equations

We now describe the approach to the Hamilton-Jacobi equations in terms of calculus of variations rather than optimal control. In order to conform to an earlier version of the notes, we will consider the initial rather than terminal value problem for the Hamilton-Jacobi equations. When the Hamiltonian is time-independent, the switch from one to another can be made by a simple change of variables $t \to T - t$. If it does depend on time, the terminal value problem is more natural in the context of optimal control but we will not try to re-write everything for that case.

Let $L(q, x)$ be a smooth function, $q, x \in \mathbb{R}^n$ called the Lagrangian. Fix two points $x, y \in \mathbb{R}^n$ and consider the class of admissible trajectories connecting these points:

$$\mathcal{A} = \{w \in C([0, t]; \mathbb{R}^n) : \ w(0) = y, \ w(t) = x\},$$

that is $w(t)$ are smooth paths that start at y at time zero, and end at x at time t. Define the functional

$$I(w) = \int_0^t L(\dot{w}(s), w(s))ds.$$

The basic problem of the calculus of variations is to find the optimal curve $w(t)$:

$$\text{find } I^* = \min_{w \in \mathcal{A}} I(w),$$

and, if possible, the optimal path $z(s) \in \mathcal{A}$ such that $I(z) = I^*$. Let us first assume that such $z(s)$ exists and deduce some of its properties.

Theorem 2.4. *(Euler-Lagrange equations) The function $z(s)$ satisfies the Euler-Lagrange equations*

$$-\frac{d}{ds}[\nabla_q L(\dot{z}(s), z(s))] + \nabla_x L(\dot{z}(s), z(s)) = 0, \quad 0 \le s \le t. \tag{2.51}$$

Proof. Let $z(t)$ be a minimizer, and $v(t)$ be a smooth function such that $v(0) = v(t) = 0$, and consider $w_\tau(s) = z(s) + \tau v(s)$. Set also $r(\tau) = I(w_\tau)$. As $z(s)$ minimizes $I(w)$ over $\mathcal{A}$ and $w_\tau \in \mathcal{A}$ for all τ, we have $r'(0) = 0$. Let us now compute $r'(\tau)$:

$$r(\tau) = \int_0^t L(\dot{z}(s) + \tau \dot{v}(s), z(s) + \tau v(s))ds,$$

so

$$r'(\tau) = \int_0^t [\nabla_q L \cdot \dot{v}(s) + \nabla_x L \cdot v(s)]ds = \int_0^t [-\frac{d}{ds}\nabla_q L + \nabla_x L] \cdot v(s)ds.$$

We integrated by parts in the second equality above, and used the fact that the boundary terms vanish since $v(0) = v(t) = 0$. Since $r'(0) = 0$ for all $v(s)$ as above, we should have

$$-\frac{d}{ds}\nabla_q L(\dot{z}(s), z(s)) + \nabla_x L(\dot{z}(s), z(s)) = 0,$$

which is (2.51). □

The above computation shows that if $z(s)$ is a minimizer then it has to satisfy the Euler-Lagrange equation (2.51). However, of course, it is possible that $z(s)$ is a critical point of $I(w)$ but not its minimum – in that case $z(s)$ also satisfies the Euler-Lagrange equations.

The Hamilton equations

There is a nice connection between the Euler-Lagrange equations and the Hamilton equations of classical mechanics. We assume that the equation

$$p = \nabla_q L(q, x) \tag{2.52}$$

can be solved uniquely as an equation for q, as a smooth function of p and x. If that is the case, we can define the Hamiltonian

$$H(p, x) = p \cdot q(p, x) - L(q(p, x), x), \tag{2.53}$$

with the function $q(p, x)$ defined implicitly by (2.52).

Let us now assume that $z(s)$ is the solution of the Euler-Lagrange equations, and set

$$p(s) = \nabla_q L(\dot{z}(s), z(s)), \tag{2.54}$$

that is,

$$\dot{z}(s) = q(p(s), z(s)). \tag{2.55}$$

Differentiating (2.53) in p_j gives

$$\frac{\partial H(p(s), z(s)}{\partial p_j} = q_j(p(s), z(s)) + \sum_{i=1}^{n} p_i(s)\frac{\partial q_i}{\partial p_j} - \sum_{i=1}^{m} \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial p_j} = q_j.$$

We used (2.54) in the last step. Using this in (2.55) gives

$$\dot{z}_j(s) = \frac{\partial H(p(s), z(s))}{\partial p_j}. \tag{2.56}$$

The Euler-Lagrange equations say that

$$\dot{p}_j(s) = \frac{\partial L}{\partial x_j}. \tag{2.57}$$

Differentiating (2.53) in x gives:

$$\frac{\partial H}{\partial x_j} = \sum_{i=1}^{n} p_i \frac{\partial q_i}{\partial x_j} - \frac{\partial L}{\partial x_j} - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial x_j} = -\frac{\partial L}{\partial x_j}.$$

Now, putting this together with (2.56)-(2.57) gives the Hamiltonian system

$$\dot{z}(s) = \nabla_p H(p(s), z(s)), \quad \dot{p}(s) = -\nabla_z H(p(s), z(s)). \tag{2.58}$$

The Legendre transform

Let us now assume that the Lagrangian does not depend on the variable x: $L = L(q)$. Then the Hamiltonian $H(p)$ is

$$H(p) = p \cdot q(p) - L(q(p)), \tag{2.59}$$

with q being the solution of

$$p = \nabla_q L(q). \tag{2.60}$$

In order to ensure that the function $q(p)$ is well-defined let us assume that the function $L(q)$ is convex and

$$\lim_{|q|\to+\infty} \frac{L(q)}{|q|} = +\infty. \tag{2.61}$$

Let us now fix p and consider the function $r(q) = p \cdot q - L(q)$. This function is concave and $r(q) \to -\infty$ as $|q| \to +\infty$. Therefore, $r(q)$ attains a unique maximum at the point where $p = \nabla L(q)$, which is exactly (2.60). Thus, we may reformulate (2.59) as

$$H(p) = \sup_q (p \cdot q - L(q)). \tag{2.62}$$

The function $H(p)$ defined by (2.62) is called the Legendre transform of $L(q)$, denoted as $H(p) = L^*(q)$.

Theorem 2.5. *Assume that the function $L(q)$ is convex and (2.61) holds, then $H(p)$ is also convex, and*

$$\lim_{|p|\to+\infty} \frac{H(p)}{|p|} = +\infty. \tag{2.63}$$

Moreover, $L(q)$ is the Legendre transform of the function H.

Proof. The function $s(p;q) = p \cdot q - L(q)$ is an affine function of p for each q fixed. Therefore, $H(p)$ is a supremum of a family of affine functions – hence, it is convex. Indeed, for any $\lambda \in (0,1)$ we have

$$\begin{aligned}
&H(\lambda p_1 + (1-\lambda)p_2) = \sup_q (\lambda p_1 + (1-\lambda)p_2 \cdot q) - L(q) \\
&= \sup_q [\lambda p_1 \cdot q - \lambda L(q) + (1-\lambda)p_2 \cdot q - (1-\lambda)L(q)] \\
&\le \sup_q [\lambda p_1 \cdot q - \lambda L(q)] + \sup_q [(1-\lambda)p_2 \cdot q - (1-\lambda)L(q)] = \lambda H(p_1) + (1-\lambda)H(p_2),
\end{aligned}$$

hence $H(p)$ is convex.

In order to see that (2.63) holds, fix $\lambda > 0$ and take $\bar{q} = \lambda p/|p|$ in the definition of $H(p)$, then $|\bar{q}| \le \lambda$, hence

$$H(p) \ge p \cdot \bar{q} - L(\bar{q}) = \lambda|p| - L(\bar{q}) \ge \lambda|p| - \sup_{|q|\le\lambda} L(q).$$

It follows that

$$\lim_{|p|\to+\infty} \frac{H(p)}{|p|} \ge \lambda,$$

for each $\lambda > 0$, thus (2.63) holds.

In order to show that $L(q)$ is actually the Legendre transform of $H(p)$, note that, for all p and q we have

$$H(p) \geq p \cdot q - L(q),$$

whence

$$L(q) \geq p \cdot q - H(p).$$

It follows that $L(q) \geq H^*(q)$. But we also have

$$H^*(q) = \sup_p [p \cdot q - H(p)] = \sup_p [p \cdot q - \sup_y [p \cdot y - L(y)]] = \sup_p \inf_y [p \cdot (q - y) + L(y)]. \quad (2.64)$$

As the function $L(q)$ is convex, for each q there exists $s(q)$ such that the graph of $L(y)$ lies above the corresponding hyperplane:

$$L(y) \geq L(q) + s \cdot (y - q).$$

Let us take $p = s(q)$ in (2.64):

$$H^*(q) \geq \inf_y [s \cdot (q - y) + L(y)] \geq L(q). \quad (2.65)$$

We conclude that $H^*(p) = L(q)$. $\square$

The Hopf-Lax formula

We now relate the variational problem that we looked at to the Hamilton-Jacobi equations. Consider the initial value problem

$$u_t + H(\nabla u) = 0, \quad t > 0, \ x \in \mathbb{R}^n, \quad (2.66)$$

with the initial data $u(0, x) = g(x)$. The initial data $g(x)$ is globally Lipschitz continuous:

$$\mathrm{Lip}(g) = \sup_{x,y \in \mathbb{R}^n} \frac{|g(x) - g(y)|}{|x - y|} < +\infty. \quad (2.67)$$

We assume that $H(p)$ is convex and satisfies the growth condition (2.63). Let us define

$$u(t, x) = \inf \left[\int_0^t L(\dot{w}(s)) ds + g(y) : \ w(0) = y, \ w(t) = x \right], \quad (2.68)$$

with the infimum taken over all C^1 functions $w(t)$ that satisfy the constraint $w(t) = x$. Here, $L(q)$ is the Legendre transform of the function $H(p)$. We will show that expression (2.68) gives a solution of the Hamilton-Jacobi equation (2.66).

Theorem 2.6. *(Hopf-Lax formula) The function $u(t, x)$ defined by (2.68) can be written as*

$$u(t, x) = \min_{y \in \mathbb{R}^n} \left[tL\left(\frac{x - y}{t} \right) + g(y) \right]. \quad (2.69)$$

Proof. First, for any $y \in \mathbb{R}^n$ we may take a "test path"

$$w(s) = y + \frac{s}{t}(x - y),$$

leading to

$$u(t,x) \leq \int_0^t L(\frac{x-y}{t})ds + g(y) = tL\left(\frac{x-y}{t}\right) + g(y).$$

As a consequence, we have

$$u(t,x) \leq \inf_{y \in \mathbb{R}^n} \left[tL\left(\frac{x-y}{t}\right) + g(y) \right].$$

On the other hand, Jensen's inequality implies that for any test path $w(s)$ we have

$$\frac{1}{t}\int_0^t L(\dot{w}(s))ds \geq L\left(\frac{1}{t}\int_0^t \dot{w}(s)ds\right).$$

Therefore,

$$\int_0^t L(\dot{w}(s))ds \geq tL\left(\frac{x-y}{t}\right),$$

where $y = w(0)$, and thus

$$u(t,x) \geq \inf_{y \in \mathbb{R}^n} \left[tL\left(\frac{x-y}{t}\right) + g(y) \right].$$

Thus, we have shown that

$$u(t,x) = \inf_{y \in \mathbb{R}^n} \left[tL\left(\frac{x-y}{t}\right) + g(y) \right].$$

The fact that the infimum in the right side is actually achieved follows from the fact that for each t and x fixed the function

$$r(y) = tL\left(\frac{x-y}{t}\right) + g(y)$$

tends to $+\infty$ as $|y| \to +\infty$. This is because $L(y)$ is super-linear at infinity, and g is globally Lipschitz. $\Box$

A formal computation of the Hamilton-Jacobi equation

Let us now show why we expect the function given by the Hopf-Lax formula to satisfy the Hamilton-Jacobi equation, assuming that it is as smooth as needed. For simplicity, assume that $x \in \mathbb{R}$. Let z be such that

$$u(t,x) = tL(\frac{x-z}{t}) + g(z).$$

Then z is determined by the condition

$$g'(z) = L'(\frac{x-z}{t}), \tag{2.70}$$

hence we have

$$u_t = L(\frac{x-z}{t}) - L'(\frac{x-z}{t})z_t - \frac{(x-z)}{t}L'(\frac{x-z}{t}) + g'(z)z_t = L(\frac{x-z}{t}) - \frac{(x-z)}{t}L'(\frac{x-z}{t}).$$

Moreover,

$$u_x = L'((x-z)/t), \tag{2.71}$$

hence the above can be written as

$$u_t = L(\frac{x-z}{t}) - u_x\frac{(x-z)}{t}. \tag{2.72}$$

On the other hand, in the definition of $H(p)$ we have

$$H(p) = \sup_{y\in\mathbb{R}}(py - L(y)) = pq - L(q),$$

with q determined by the relation $p = L'(q)$. Therefore,

$$H(u_x) = u_x q - L(q),$$

with q such that $u_x = L'(q)$. But (2.71) implies that then $q = (x-z)/t$, and (2.72) is nothing but the Hamilton-Jacobi equation

$$u_t + H(u_x) = 0.$$

Exercise. Check the following fact. Let $X(t)$ and $P(t)$ be the solution of the Hamiltonian system

$$\dot{X}(t) = \nabla_p H(X(t), P(t)), \quad \dot{P}(t) = -\nabla_x H(X(t), P(t)), \tag{2.73}$$

with the initial condition $X(T) = x$, $P(T) = \nabla g(x)$. Check that, as long as the solution of the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + H(x, \nabla u) = 0, \quad u(T, x) = g(x), \tag{2.74}$$

remains smooth, we have $u(t,x) = g(X(t))$ and $\nabla u(t,x) = P(t)$. Relate this to drift in the Fokker-Planck equation in the mean-field game system (5.1).

The rigorous derivation of the Hamilton-Jacobi equation

Let us now verify that the Hopf-Lax formula is Lipschitz continuous.

Lemma 2.7. *Let $u(t,x)$ be defined by (2.69). Then the function $u(t,x)$ is Lipschitz continuous in x for $t \geq 0$ and $x \in \mathbb{R}^n$, and $u(t,x) \to g(x)$ as $t \to 0$.*

Proof. Take $x_1, x_2 \in \mathbb{R}^n$, and choose y so that

$$u(t,x_1) = tL(\frac{x_1 - y}{t}) + g(y),$$

then, choosing $z = x_2 - x_1 + y$ below, gives

$$\begin{aligned} &u(t,x_1) - u(t,x_2) = \min_{z\in\mathbb{R}^n}\left[tL\left(\frac{x_2 - z}{t}\right) + g(z)\right] - tL(\frac{x_1 - y}{t}) - g(y) \\ &\leq g(x_2 - x_1 + y) - g(y) \leq Lip(g)|x_1 - x_2|. \end{aligned}$$

Switching the roles of x_1 and x_2 gives Lipschitz continuity in x:

$$|u(t,x_1) - u(t,x_2)| \leq Lip(g)|x_1 - x_2|.$$

In order to verify the initial condition, note that choosing $y = x$ gives

$$u(t,x) \leq tL(0) + g(x), \tag{2.75}$$

but we also have

$$\begin{aligned} &u(t,x) = \min_y\left[tL\left(\frac{x-y}{t}\right) + g(y)\right] \geq \min_y\left[tL\left(\frac{x-y}{t}\right) + g(x) - Lip(g)|x-y|\right] \\ &= g(x) + \min_z[tL(z) - Lip(g)t|z|] = g(x) + t\min_z[L(z) - Lip(g)|z|]. \end{aligned}$$

once again, as $L(z)$ grows super-linearly at infinity, we have

$$\min_z[L(z) - Lip(g)|z|] > -\infty,$$

hence

$$u(t,x) \geq g(x) - Ct. \tag{2.76}$$

We conclude that $u(t,x) \to g(x)$ as $t \to 0$. $\Box$

In order to show that $u(t,x)$ is Lipschitz continuous in time, we need the following lemma (which is essentially a version of the dynamic programming principle).

Lemma 2.8. *For each $x \in \mathbb{R}^n$, and $0 \leq s < t$ we have*

$$u(t,x) = \min_{y\in\mathbb{R}^n}\left[(t-s)L\left(\frac{x-y}{t-s}\right) + u(s,y)\right]. \tag{2.77}$$

Proof. Choose z so that

$$u(s,y) = sL(\frac{y-z}{s}) + g(z).$$

Let us write

$$\frac{x-z}{t} = (1 - \frac{s}{t})\frac{x-y}{t-s} + \frac{s}{t}\frac{y-z}{s}.$$

As L is convex, it follows that

$$
\begin{aligned}
u(t,x) &\le tL(\frac{x-z}{t}) + g(z) \le t(1-\frac{s}{t})L(\frac{x-y}{t-s}) + sL(\frac{y-z}{s}) + g(z) \\
&= (t-s)L(\frac{x-y}{t-s}) + sL(\frac{y-z}{s}) + g(z) = (t-s)L(\frac{x-y}{t-s}) + u(s,y),
\end{aligned}
$$

and thus

$$
u(t,x) \le \inf_{y\in\mathbb{R}^n}\left[(t-s)L(\frac{x-y}{t-s}) + u(s,y)\right].
$$

As the function $u(s,y)$ is actually continuous in y (this follows from Lemma 2.7), and $|u(s,y)|$ grows not faster than linearly at infinity (that follows from (2.75)-(2.76)), the infimum in the right side is actually attained:

$$
u(t,x) \le \min_{y\in\mathbb{R}^n}\left[(t-s)L(\frac{x-y}{t-s}) + u(s,y)\right].
$$

In order to show the opposite inequality, choose z so that

$$
u(t,x) = tL(\frac{x-z}{t}) + g(z),
$$

and set

$$
y = \frac{s}{t}x + (1-\frac{s}{t})z.
$$

Then, we have

$$
\frac{x-y}{t-s} = \frac{x-z}{t} = \frac{y-z}{s},
$$

hence

$$
(t-s)L(\frac{x-y}{t-s}) + u(s,y) \le (t-s)L(\frac{x-z}{t}) + sL(\frac{y-z}{s}) + g(z) = tL(\frac{x-z}{t}) + g(z) = u(t,x).
$$

This proves (2.77). □

Lemma 2.9. *The function $u(t,x)$ defined by (2.69) is Lipschitz continuous in t for $t \ge 0$ and $x \in \mathbb{R}^n$.*

Proof. Combining the ideas in the proof of Lemma 2.7 (see (2.75)-(2.76)) with the result of Lemma 2.8 gives

$$
u(s,x) - C(t-s) \le u(t,x) \le u(s,x) + C(t-s),
$$

and we are done. □

Since the function $u(t,x)$ is Lipschitz in t and x, it is differentiable almost everywhere.

Theorem 2.10. *The function $u(t,x)$ defined by (2.69) is Lipschitz continuous in t and x, differentiable almost everywhere and solves the initial value problem*

$$
u_t + H(\nabla u) = 0, \quad t > 0,\ x \in \mathbb{R}^n, \tag{2.78}
$$

with $u(0,x) = g(x)$.

Proof. It remains only to verify that at the points (t, x) where both u_t and ∇u exist, the Hamilton-Jacobi equation (2.78) is satisfied. Fix $q \in \mathbb{R}^n$, $h > 0$, then we have, according to Lemma 2.8:

$$u(x + hq, t + h) = \min_{y\in\mathbb{R}^n} \left[hL(\frac{x + hq - y}{h}) + u(t, y) \right] \leq hL(q) + u(t, x).$$

It follows that

$$u_t(t, x) + q \cdot \nabla u(t, x) \leq L(q),$$

for all $q \in \mathbb{R}^n$. Therefore, we have

$$u_t(t, x) + H(\nabla u(t, x)) = u_t(t, x) + \max_{q\in\mathbb{R}^n}(q \cdot \nabla u(t, x) - L(q)) \leq 0. \tag{2.79}$$

Next, we show the opposite inequality. Choose z so that

$$u(t, x) = tL(\frac{x - z}{t}) + g(z).$$

Given $h > 0$, set

$$y = \frac{t - h}{t}x + (1 - \frac{t - h}{t})z = x - h\frac{(x - z)}{t}, \tag{2.80}$$

so that

$$\frac{x - z}{t} = \frac{y - z}{t - h}.$$

We have

$$u(t, x) - u(t - h, y) \geq tL(\frac{x - z}{t}) + g(z) - \left[(t - h)L(\frac{y - z}{t - h}) + g(z) \right] = hL(\frac{x - z}{t}).$$

Keeping in mind expression (2.80), and letting $h \to 0$ gives

$$u_t(t, x) + \frac{1}{t}(x - z) \cdot \nabla u(t, x) \geq L(\frac{x - z}{t}).$$

It follows that

$$u_t(t, x) + H(\nabla u(t, x)) = u_t(t, x) + \max_{q\in\mathbb{R}^n}(q \cdot \nabla u(t, x) - L(q)) \geq 0,$$

which, together with (2.79) finishes the proof. □

2.8 Viscosity solutions for Hamilton-Jacobi equations

We will now consider solutions of the Cauchy problem for the Hamilton-Jacobi equations

$$\begin{aligned} &u_t + H(\nabla u, x) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n, \\ &u(0, x) = g(x). \end{aligned} \tag{2.81}$$

The idea is to consider solutions of the regularized parabolic problem

$$\begin{aligned} &u^\varepsilon_t + H(\nabla u^\varepsilon, x) = \varepsilon\Delta u^\varepsilon, \\ &u^\varepsilon(0, x) = g(x). \end{aligned} \tag{2.82}$$

The idea is to show that for each $\varepsilon > 0$ the problem (2.81) admits a regular solution, and then pass to the limit $\varepsilon \to 0$. The difficulty is that as $\varepsilon \to 0$ the regularizing effect of the Laplacian is less and less, so $u^\varepsilon(t,x)$ are less and less regular, and it is not clear that the limit of $u^\varepsilon(t,x)$, if it exists, is regular, and in which sense it would satisfy the Hamilton-Jacobi equation (2.82).

Let us for the moment assume that for some sequence $\varepsilon_n \to 0$ (2.81) has a smooth solution $u_n(t,x) = u^{\varepsilon_n}(t,x)$, and that $u_n(t,x) \to u(t,x)$ locally uniformly in $\mathbb{R}^n \times [0,+\infty)$. Let us take a smooth test function v and suppose that $u(t,x) - v(t,x)$ has a strict local minimum at some point (t_0,x_0). Then $u(t,x) - v(t,x) > u(t_0,x_0) - v(t_0,x_0)$ in some neighborhood B of (t_0,x_0). The functions $u_n(t,x) - v(t,x)$ have to attain a local maximum inside B when n is sufficiently large as well – simply because we have

$$\max_{\partial B}(u_n(t,x) - v(t,x)) < u_n(t_0,x_0) - v(t_0,x_0).$$

Hence, $u_n(t,x) - v(t,x)$ attains a maximum in B. Now, if we let the radius of B go to zero, we get a sequence of points $(t_n,x_n) \to (t_0,x_0)$ such that $u_n(t,x) - v(t,x)$ has a local maximum at (t_n,x_n). We deduce that $\nabla u_n(t_n,x_n) = \nabla v(t_n,x_n)$, $u_{n,t}(t_n,x_n) = v_t(t_n,x_n)$ and

$$-\Delta u_n(t_n,x_n) \geq -\Delta v(t_n,x_n).$$

It follows that

$$\begin{aligned} &v_t(t_n,x_n) + H(\nabla v(t_n,x_n),x_n) = u_{n,t}(t_n,x_n) + H(\nabla u_n(t_n,x_n),x_n) = \varepsilon\Delta u_n(t_n,x_n) \\ &\leq \varepsilon\Delta v(t_n,x_n). \end{aligned} \tag{2.83}$$

The function $v(t,x)$ is smooth, so we may let $\varepsilon \to 0$ in (2.83) to conclude that

$$v_t(t_0,x_0) + H(\nabla v(t_0,x_0),x_0) \leq 0. \tag{2.84}$$

Inequality (2.84) should hold for any smooth function $v(t,x)$ such that $u(t,x) - v(t,x)$ attains a local maximum at (t_0,x_0). Similarly, if $u - v$ attains a local minimum at (t_0,x_0) then we should have

$$v_t(t_0,x_0) + H(\nabla v(t_0,x_0),x_0) \geq 0. \tag{2.85}$$

The above argument assumed that solutions of the regularized parabolic problem exist and have a limit $u(t,x)$. Let us now instead take the inequalities (2.84) and (2.85) as the starting point and define the appropriate solution of the Hamilton-Jacobi equation purely in their terms, forgetting everything about the parabolic problem.

Definition 2.11. *A bounded uniformly continuous function $u(t,x)$ is a viscosity solution of the Cauchy problem (2.81) for the Hamilton-Jacobi equation if $u(0,x) = g(x)$ for all $x \in \mathbb{R}^n$, and for each $v \in C^\infty([0,\infty) \times \mathbb{R}^n)$ such that $u - v$ has a local maximum at a point (t_0,x_0) with $t_0 > 0$, we have*

$$v_t(t_0,x_0) + H(\nabla v(t_0,x_0),x_n) \leq 0, \tag{2.86}$$

while if $u - v$ attains a local minimum at a point (t_0,x_0) with $t_0 > 0$, we have

$$v_t(t_0,x_0) + H(\nabla v(t_0,x_0),x_n) \geq 0, \tag{2.87}$$

We will verify that these two conditions are reasonable in the following sense.

Theorem 2.12. *(Consistency) Let $u(t,x)$ be a viscosity solution of the Cauchy problem*

$$\begin{aligned} &u_t + H(\nabla u, x) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n, \\ &u(0,x) = g(x). \end{aligned} \tag{2.88}$$

Assume that u is differentiable at some point (t_0, x_0) with $t_0 > 0$, then

$$u_t(t_0, x_0) + H(\nabla u(t_0, x_0), x_0) = 0. \tag{2.89}$$

We begin the proof with the following lemma.

Lemma 2.13. *Assume that $u(x)$, $x \in \mathbb{R}^n$ is a continuous function and $u(x)$ is differentiable at x_0. Then there exists a $C^1(\mathbb{R}^n)$ function $q(x)$ such that $u(x_0) = v(x_0)$ and $u - v$ has a strict local maximum at x_0.*

Proof of Lemma. Let us set

$$v(x) = u(x + x_0) - u(x_0) - x \cdot \nabla u(x_0),$$

so that $v(0) = 0$, $\nabla v(0) = 0$. It follows that $v(x) = |x|\rho(x)$, where the function $\rho(x)$ is continuous, and $\rho(0) = 0$. Set

$$p(r) = \max_{x \in B(0,r)} \rho(x),$$

then $p(r)$ is continuous, non-decreasing and $p(0) = 0$. Finally, define

$$w(x) = |x|^2 + \int_{|x|}^{2|x|} p(r)dr.$$

Then $w \in C^1(\mathbb{R}^n)$, and

$$|w(x)| \leq |x|^2 + |x|p(2|x|),$$

which means that $w(0) = 0$ and $\nabla w(0) = 0$. However, we have

$$\begin{aligned} v(x) - w(x) &= |x|\rho(x) - |x|^2 - \int_{|x|}^{2|x|} p(r)dr \leq |x|p(|x|) - |x|^2 - \int_{|x|}^{2|x|} p(r)dr \\ &\leq -|x|^2 < 0 = v(0) - w(0). \end{aligned}$$

Therefore, the function $v(x) - w(x)$ attains its local maximum at $x = 0$, which means that we can take

$$q(x) = w(x - x_0) + u(x_0) + (x - x_0) \cdot \nabla u(x_0),$$

proving Lemma 2.13. □

Proof of Theorem 2.12. Note that if $u(t,x)$ were C^∞ (rather than just differentiable at (t_0, x_0)), we could take u itself as a test function in the definition of the viscosity solution, and conclude that hence both

$$u_t(t_0, x_0) + H(\nabla u(t_0, x_0), x_0) \leq 0,$$

and

$$u_t(t_0, x_0) + H(\nabla u(t_0, x_0), x_0) \geq 0,$$

giving the result. Hence, what we need to do is replace u by a smooth test function without changing u_t and ∇u too much. Lemma 2.13 implies that there exists a C^1 function v such that $u - v$ has a strict maximum at (x_0, t_0). Next, let $v_\varepsilon(t, x)$ be

$$v_\varepsilon(t, x) = \frac{1}{\varepsilon^{n+1}} \int \chi(\frac{t-s}{\varepsilon}, \frac{x-y}{\varepsilon}) v(s, y) ds dy.$$

Here the function $\chi(t, x) \in C^\infty(\mathbb{R}^{n+1})$ is chosen so that $\chi(t, x) \geq 0$, and

$$\int \chi(t, x) dt dx = 1.$$

Then the functions $v_\varepsilon \in C^\infty$ for all $\varepsilon > 0$, and $v_\varepsilon \to v$, $v_{\varepsilon,t} \to v_t$, $\nabla v_\varepsilon \to \nabla v$, all locally uniformly near (t_0, x_0). It follows that $u(t, x) - v_\varepsilon(t, x)$ has a strict local maximum at some point $(t_\varepsilon, x_\varepsilon)$ with $(t_\varepsilon, x_\varepsilon) \to (t_0, x_0)$ as $\varepsilon \to 0$. The definition of the viscosity solution implies that

$$v_{\varepsilon,t}(t_\varepsilon, x_\varepsilon) + H(\nabla v_\varepsilon(t_\varepsilon, x_\varepsilon), x_\varepsilon) \leq 0.$$

Passing to the limit $\varepsilon \to 0$ gives

$$v_t(t_0, x_0) + H(\nabla v(t_0, x_0), x_0) \leq 0.$$

Since $u(t, x)$ is differentiable at (t_0, x_0) and $u - v$ attains a local maximum at (t_0, x_0), we have

$$u_t(t_0, x_0) = v_t(t_0, x_0), \quad \nabla u(t_0, x_0) = \nabla v(t_0, x_0),$$

hence

$$u_t(t_0, x_0) + H(\nabla u(t_0, x_0), x_0) \leq 0.$$

Similarly, we can prove that

$$v_t(t_0, x_0) + H(\nabla v(t_0, x_0), x_0) = 0,$$

and we are done. □

Viscosity solution (if it exists) is unique.

Theorem 2.14. *(Uniqueness) There exists at most one viscosity solution of the Cauchy problem*

$$\begin{aligned} &u_t + H(\nabla u, x) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n, \\ &u(0, x) = g(x). \end{aligned} \tag{2.90}$$

Hopf-Lax formula as a viscosity solution

Let us now show that the Hopf-Lax formula gives a viscosity solution for the Cauchy problem

$$u_t + H(\nabla u) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n, \qquad (2.91)$$
$$u(0,x) = g(x),$$

if $H(p)$ is convex,

$$\lim_{|p| \to +\infty} \frac{H(p)}{|p|} = +\infty,$$

and $g(x)$ is bounded and Lipschitz continuous. Let L be the Legendre transform of H:

$$L(q) = \sup_{p \in \mathbb{R}^n} (p \cdot q - H(p)),$$

and set

$$u(t,x) = \min_{y \in \mathbb{R}^n} \left[tL(\frac{x-y}{t}) + g(y) \right]. \qquad (2.92)$$

Let us show that $u(t,x)$ is the viscosity solution of (2.91). We already know that $u(t,x)$ defined by (2.92) is Lipschitz continuous in t and x.

Take $v \in C^\infty$ and assume that $u - v$ has a local maximum at (t_0, x_0). Then, we have

$$u(t_0, x_0) = \min_{x \in \mathbb{R}^n} \left[(t_0 - t)L(\frac{x_0 - x}{t_0 - t}) + u(t,x) \right] \leq (t_0 - t)L(\frac{x_0 - x}{t_0 - t}) + u(t,x),$$

for all $0 \leq t < t_0$, and $x \in \mathbb{R}^n$. Since $u - v$ has a local maximum at (t_0, x_0), we also have

$$u(t,x) - v(t,x) \leq u(t_0, x_0) - v(t_0, x_0),$$

for t, x close to t_0, x_0. Hence,

$$v(t_0, x_0) - v(t,x) \leq u(t_0, x_0) - u(t,x) \leq (t_0 - t)L(\frac{x_0 - x}{t_0 - t}).$$

Let us use this relation for $t = t_0 - h$ and $x = x_0 - hq$, with some $h > 0$ fixed, and $q \in \mathbb{R}^n$. We get

$$v(t_0, x_0) - v(t_0 - h, x_0 - hq) \leq hL(q).$$

Passing to the limit $h \to 0$ gives

$$v_t + q \cdot \nabla v(t_0, x_0) \leq L(q).$$

As this is true for all q, we deduce that

$$v_t(t_0, x_0) + H(\nabla v(t_0, x_0)) \leq 0.$$

Next, suppose that $u - v$ attains a local minimum at (t_0, x_0). We will show that

$$v_t(t_0, x_0) + H(\nabla v(t_0, x_0)) \geq 0.$$

If this is false, then there exists some $\theta > 0$ so that

$$v_t(t,x) + H(\nabla v(t,x)) \leq -\theta < 0,$$

for all t and x close to t_0, x_0. It follows that

$$v_t(t,x) + q \cdot \nabla u(t,x) - L(q) \leq -\theta, \tag{2.93}$$

for all such t, x and all $q \in \mathbb{R}^n$. Now, for $h > 0$ small enough there exists x_1 close to x_0 so that

$$u(t_0, x_0) = hL(\frac{x_0 - x_1}{h}) + u(t_0 - h, x_1).$$

Let us look at (2.93) with $q = (x_0 - x_1)/h$, then we get

$$v(x_0, t_0) - v(t - h, x_1) \leq h\left(L(\frac{x_0 - x_1}{h}) - \theta\right).$$

But that means

$$v(x_0, t_0) - v(t - h, x_1) < u(x_0, t_0) - u(t - h, x_1).$$

This is a contradiction to $u - v$ attaining a local minimum at (t_0, x_0). $\square$

3 Nonatomic games

We return to the Cardaliaguet notes.

Before starting the analysis of differential games with a large number of players, it is helpful to look at this question for classical games. The general framework is as follows: let N be a (large) number of players. We assume that the players are symmetric (identical), so that the set Q of available strategies is the same for all players. We denote by $F_i^N = F_i^N(x_1, \ldots, x_N)$ the payoff (or the cost) of player $i \in \{1, \ldots, N\}$ given the "all-players" state $(x_1, \ldots, x_N)$. The symmetry assumption means that

$$F^N_{\sigma(i)}(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = F_i(x_1, \ldots, x_N)$$

for all permutations σ on $\{1, \ldots, N\}$. Our goal is to analyze the behavior of the Nash equilibria for this game as $N \to +\infty$.

For this we first recall the notion of Nash equilibria. In order to proceed with the analysis of large population games, we describe next the limit of maps of many variables. Then we explain the limit, as the number of players tends to infinity, of Nash equilibria in pure, and then in mixed, strategies. This is how the mean-field game equation comes about. We finally discuss the uniqueness of the solution of the limit equation and present some examples.

3.1 Nash equilibria in classical differential games

Here, we introduce the notion of Nash equilibria in one-shot games. Let $Q_1, \ldots, Q_N$ be compact metric spaces – the elements of Q_i are the possible strategies of player i, and $J_1, \ldots, J_N$ be continuous real valued functions on $\prod_{i=1}^N Q_i$.

Definition 3.1. *A Nash equilibrium in pure strategies is a* $N-$*tuple* $(\bar{s}_1, \ldots, \bar{s}_N) \in \prod_{i=1}^N Q_i$ *such that, for any* $i = 1, \ldots, N$,

$$J_i(\bar{s}_1, \ldots, \bar{s}_N) \le J_i\left(s_i, (\bar{s}_j)_{j \neq i}\right) \qquad \forall s_i \in Q_i \ .$$

In other words, a Nash equilibrium is a set of strategies $\bar{s}_1, \ldots, \bar{s}_N$ such that it is "expensive" for a player i to deviate from $\bar{s}_i$ provided that all other players uses strategies $\bar{s}_k$, $k \neq i$. Let us consider a couple of examples.

Example 3.2. Consider two players who can set prices p_1 and p_2, with $0 \le p_1, p_2 \le 1$, and sell $x_1(p_1, p_2)$ and $x_2(p_1, p_2)$ units respectively, with

$$x_2(p_1, p_2) = \frac{2}{3}(p_1 - p_2), \text{ if } p_1 \ge p_2, \text{ and } x_2(p_1, p_2) = 0 \text{ if } p_1 < p_2,$$

and $x_1(p_1, p_1) = 1 - x_2(p_1, p_2)$. The profit of the two players is

$$u_1(p_1, p_2) = p_1 x_1(p_1, p_2), \quad u_2(p_1, p_2) = p_2 x_2(p_1, p_2).$$

Then, given the strategy p_2, for the first player the optimization problem is to maximize the function $u_1 = p_1 x_1$, with

$$x_1 = 1 - \frac{2}{3}(p_1 - p_2).$$

A simple computation shows that the optimal value of p_1 (again, given p_2) is

$$\tilde{p}_1(p_2) = \min(1, \frac{3}{4} + \frac{p_2}{2}).$$

The second player optimizes $u_2 = p_2 x_2$, subject to the constraint $x_2 = (2/3)(p_1 - p_2)$, so the optimal price for him (given p_1) is $\tilde{p}_2(p_1) = p_1/2$. Then the unique Nash equilibrium is $\bar{p}_1 = 1$ and $\bar{p}_2 = 1/2$.

Example 3.3. Let us look at a similar example but with slightly different constraints. Again, the profits of the two players are $u_1(p_1, p_2) = p_1 x_1(p_1, p_2)$ and $u_2(p_1, p_2) = p_2 x_2(p_1, p_2)$, with $x_1(p_1, p_1) = 1 - x_2(p_1, p_2)$. However, we now have

$$p_1 = p_2 + l(x_2), \quad l(x) = \frac{x - 1/2}{\varepsilon}, \text{ if } x \ge 1/2, \text{ and } l(x) = 0 \text{ if } 0 \le x \le 1/2.$$

Then one can directly check that a pure Nash equilibrium does not exist when $\varepsilon > 0$ is sufficiently small, according to some MIT slides that use some jargon.

Example 3.4. Consider the symmetric setting where $Q_1 = Q_2 = \mathbb{T}^1$, and there is a function $F(x_1, x_2)$ so that $J_1(x_1, x_2) = F(x_1, x_2)$, $J_2(x_1, x_2) = F(x_2, x_1)$. Then a point (y_1, y_2) is a pure Nash equilibrium if

$$\frac{\partial F}{\partial y_1}(y_1, y_2) = 0 \text{ and } \frac{\partial F}{\partial y_1}(y_2, y_1) = 0. \tag{3.1}$$

It is easy to construct a function F such that (3.1) has no solutions – the only requirement on the function $\partial F/\partial x_1$ is that F is periodic and

$$\int_0^1 \frac{\partial F(x_1,x_2)}{\partial x_1} dx_1 = 0 \text{ for all } 0 \le x_2 \le 1,$$

so the requirement that its zero set contains no two points symmetric with respect to the line $x_1 = x_2$ can be satisfied, and then (3.1) has no solutions.

Thus, Nash equilibria in pure strategies do not necessarily exist and we have to introduce the notion of mixed strategies – this means that each player uses a family of strategies with a certain probability distribution. Let us denote by $\mathcal{P}(Q_i)$ the space of all Borel probability measures on Q_i. A mixed strategy of player i will be an element of $\mathcal{P}(Q_i)$. The set $\mathcal{P}(Q)$ is endowed with the weak-* topology: a sequence m_N in $\mathcal{P}(Q)$ converges to $m \in \mathcal{P}(Q)$ if

$$\lim_{N\to\infty} \int_Q \varphi(x) dm_N(x) = \int_Q \varphi(x) dm(x) \qquad \forall \varphi \in \mathcal{C}(Q) \ .$$

Recall that $\mathcal{P}(Q)$ is a compact metric space for this topology, which can be metrized by the Kantorowich-Rubinstein distance:

$$d_1(\mu,\nu) = \sup\{ \int_Q f d(\mu - \nu) : \ \|f\|_{Lip(Q)} \le 1 \text{ and } \sup_{x\in Q} |f(x)| \le 1.\} \ .$$

Alternatively, this distance can be stated in terms of optimal transportation:

$$d_1(\mu,\nu) = \inf_M \int_{Q\times Q} d(x,y) dM(x,y),$$

with the infimum taken over all probability measures $dM(x,y)$ on $Q\times Q$ such that the marginals of $M(x,y)$ in x and y are μ and ν, respectively.

Definition 3.5. *A Nash equilibrium in mixed strategies is an $N-$tuple $(\bar\pi_1,\dots,\bar\pi_N) \in \prod_{i=1}^N \mathcal{P}(Q_i)$ such that, for any $i = 1,\dots,N$,*

$$J_i(\bar\pi_1,\dots,\bar\pi_N) \le J_i\left((\bar\pi_j)_{j\neq i}, \pi_i\right) \qquad \forall \pi_i \in \mathcal{P}(Q_i) \ . \tag{3.2}$$

where, with some abuse of notation, we set

$$J_i(\pi_1,\dots,\pi_N) = \int_{Q_1\times\ldots\times Q_N} J_i(s_1,\dots,s_N) d\pi_1(s_1)\dots d\pi_N(s_N) \ .$$

Theorem 3.6 (Nash (1950), Glicksberg (1952))**.** *Under the above assumptions, there exists at least one equilibrium point in mixed strategies.*

Proof. Consider the best response map $\mathcal{R}_i : X := \prod_{j=1}^N \mathcal{P}(Q_i) \to 2^{\mathcal{P}(S_i)}$ of player i:

$$\mathcal{R}_i(\pi_1,\dots,\pi_N) = \Big\{ \pi \in \mathcal{P}(Q_i) \ , \ J_i((\pi_j)_{j\neq i}, \pi) = \min_{\pi' \in \mathcal{P}(S_i)} J_i((\pi_j)_{j\neq i}, \pi') \Big\}, \tag{3.3}$$

and define $\phi(\pi_1, \ldots, \pi_N) = \prod_{i=1}^N \mathcal{R}_i(\pi_1, \ldots, \pi_N) : X \to 2^X$. Then, any fixed point x of ϕ such that $x \in \phi(x)$ is a Nash equilibrium of mixed strategies.

Existence of such fixed point is established using Fan's fixed point Theorem [55], which is an infinite-dimensional version of the Kakutani theorem. It says the following. Let X be a non-empty, compact and convex subset of a locally convex topological vector space. We say that a set-valued function $\phi : X \to 2^X$ is upper-semicontinuous if for every open set $W \subset X$, the set $\{x \in X : \ \phi(x) \subseteq W\}$ is open in X. Equivalently, for every closed set $H \subset X$, the set $\{x \in X : \ \phi(x) \cap H \neq \emptyset\}$ is closed in X. Assume also that $\phi(x)$ is non-empty, compact and convex for all $x \in X$. Then ϕ has a fixed point: $\exists \bar{x} \in X$ with $\bar{x} \in \phi(\bar{x})$.

Note that in our setting ϕ is upper semicontinuous. Indeed, let $W \subset X$ be an open set and take $x = (\pi_1, \ldots, \pi_n) \in X$ such that $\phi(x) \in W$. Then for $x' = (\pi_1', \ldots, \pi_n')$ sufficiently close to x, the minimizers in (3.3) for π_j', $j \neq i$ fixed, will be close to the minimizers corresponding to π_j, $j \neq i$ fixed, so that $\phi(x') \in W$. It is also easy to see that the values $\phi(x)$ are compact, convex and non-empty. Therefore, ϕ has a fixed point, which is a Nash equilibrium in mixed strategies by the definition of ϕ. $\square$

Let us now consider the special case where the game is symmetric. Namely, we assume that, for all $i \in \{1, \ldots, N\}$, $Q_i = Q$ and $J_i(s_1, \ldots, s_N) = J_{\theta(s_i)}(s_{\theta(1)}, \ldots, s_{\theta(N)})$ for all i and all permutations θ on $\{1, \ldots, N\}$.

Theorem 3.7 (Symmetric games)**.** *If the game is symmetric, then there is an equilibrium of the form $(\bar{\pi}, \ldots, \bar{\pi})$, where $\bar{\pi} \in \mathcal{P}(Q)$ is a mixed strategy.*

Proof. Let $X = \mathcal{P}(Q)$ and $\mathcal{R} : X \to 2^X$ be the set-valued map defined by

$$\mathcal{R}(\pi) = \left\{ \sigma \in X \ , \ J_1(\sigma, \pi, \ldots, \pi) = \min_{\sigma' \in X} J_1(\sigma', \pi, \ldots, \pi) \right\} .$$

Then $\mathcal{R}$ is upper semicontinuous with nonempty convex compact values. By Fan's fixed point Theorem, it has a fixed point $\bar{\pi}$ and, from the symmetry of the game, the $N-$tuple $(\bar{\pi}, \ldots, \bar{\pi})$ is a Nash equilibrium. $\square$

3.2 Symmetric functions of many variables

Let Q be a compact metric space and $u_N : Q^N \to \mathbb{R}$ be a symmetric function:

$$u_N(x_1, \ldots, x_N) = u_N(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \quad \text{for any permutation } \sigma \text{ on } \{1, \ldots, n\}.$$

Our aim is to define a limit for u_N – note that the number of unknowns depends on N also, so something slightly non-standard needs to be done. The idea is to associate to the points $x_1, \ldots, x_N$ the measure

$$m_X^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}.$$

Next, we interpret $u_N(X)$ as the value of a certain functional on m_X^N. To this end, we make the following two assumptions on u_N. First, a uniform bound: there exists $C > 0$ so that

$$\|u_N\|_{L^\infty(Q)} \leq C \tag{3.4}$$

Second, uniform continuity: there is a modulus of continuity ω independent of n such that

$$|u_N(X) - u_N(Y)| \leq \omega(d_1(m_X^N, m_Y^N)) \qquad \forall X, Y \in Q^N,\ \forall N \in \mathbb{N}. \tag{3.5}$$

Under these assumptions, define the maps $U^N : \mathcal{P}(Q) \to \mathbb{R}$ by

$$U^N(m) = \inf_{X \in Q^N} \left\{ u_N(X) + \omega(d_1(m_X^N, m)) \right\} \qquad \forall m \in \mathcal{P}(Q)\ .$$

Then, by assumption (3.5), we have

$$U^N(m_X^N) = \inf_{Y \in Q^N} \left\{ u_N(Y) + \omega(d_1(m_Y^N, m_X^N)) \right\} = u_N(X),\ \text{for any } X \in Q^N.$$

With this interpretation, instead of talking about the convergence of the functions u_N that are defined on different spaces Q_N that depend on N, we can talk about convergence of the functionals U^N that are all defined on $\mathcal{P}(Q)$.

Theorem 3.8. *If u_N are symmetric and satisfy (3.4) and (3.5), then there is a subsequence u_{N_k} of u_N and a continuous map $U : \mathcal{P}(Q) \to \mathbb{R}$ such that*

$$\lim_{k \to +\infty} \sup_{X \in Q^{N_k}} |u_{N_k}(X) - U(m_X^{N_k})| = 0\ .$$

Proof of Theorem 3.8. Without loss of generality we can assume that the modulus ω is concave. Let us show that the U^N have ω for modulus of continuity on $\mathcal{P}(Q)$: if $m_1, m_2 \in \mathcal{P}(Q)$ and if $X \in Q^N$ is $\varepsilon-$optimal in the definition of $U^N(m_2)$:

$$u_N(x) + \omega(d_1(m_X^N, m_2)) \leq U^N(m_2) + \varepsilon,$$

then we have

$$\begin{array}{rcl} U^N(m_1) & \leq & u_N(X) + \omega(d_1(m_X^N, m_1)) \leq u_N(X) + \omega(d_1(m_X^N, m_2) + d_1(m_1, m_2)) \\ & \leq & U^N(m_2) + \varepsilon + \omega(d_1(m_X^N, m_2) + d_1(m_1, m_2)) - \omega(d_1(m_X^N, m_2)) \\ & \leq & U^N(m_2) + \omega(d_1(m_1, m_2)) + \varepsilon, \end{array}$$

because ω is concave. Hence the family U^N are equicontinuous on the compact set $\mathcal{P}(Q)$ and uniformly bounded. We complete the proof thanks to the Ascoli Theorem. □

Remark 3.9. Some uniform continuity condition is needed: for instance if Q is a compact subset of $\mathbb{R}^d$ and $u_N(X) = \max_i |x_i|$, then u_N "converges" to $U(m) = \sup_{x \in spt(m)} |x|$ which is not continuous. Of course the convergence is not uniform.

Remark 3.10. If Q is a compact subset of some finite dimensional space $\mathbb{R}^d$, a typical condition which ensures (3.5) is the existence of a constant $C > 0$, independent of N, such that

$$\sup_{i=1,\dots,N} \|D_{x_i} u_N\|_\infty \leq \frac{C}{N} \qquad \forall N.$$

3.3 Limits of Nash equilibria in pure strategies

Let us assume is that the payoffs $F_1^N, \dots, F_N^N$ of the players are symmetric. In particular, under suitable bounds and uniform continuity, we know from Theorem 3.8 that F_i^N have a limit, which has the form $F(x,m)$. Here, the dependence on x is to keep track of the dependence on i of the function F_i^N. So the payoffs of the players are very close to the form

$$F(x_1, \frac{1}{N-1}\sum_{j\geq 2}\delta_{x_j}), \dots, F(x_N, \frac{1}{N-1}\sum_{j\leq N-1}\delta_{x_j}).$$

In order to keep the presentation as simple as possible, we suppose that the payoffs already have this form. That is, we suppose that there is a continuous map $F : Q \times \mathcal{P}(Q) \to \mathbb{R}$ such that, for any $i \in \{1, \dots, N\}$

$$F_i^N(x_1, \dots, x_N) = F\Big(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\Big) \qquad \forall (x_1, \dots, x_N) \in Q^N \ .$$

Let us recall that a pure Nash equilibrium for the game $(F_1^N, \dots, F_N^N)$ is $(\bar{x}_1^N, \dots, \bar{x}_N^N) \in Q^N$ such that

$$F_i^N(\bar{x}_1^N, \dots, \bar{x}_{i-1}^N, y_i, \bar{x}_{i+1}^N, \dots, \bar{x}_N^N) \geq F_i^N(\bar{x}_1^N, \dots, \bar{x}_N^N) \qquad \forall y_i \in Q \ .$$

We set

$$\bar{X}^N = (\bar{x}_1^N, \dots, \bar{x}_N^N) \qquad \text{and} \qquad m_{\bar{X}^N}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\bar{x}_i^N} \ .$$

Theorem 3.11. *Assume that $\bar{X}^N = (\bar{x}_1^N, \dots, \bar{x}_N^N)$ is a Nash equilibrium in pure strategies for the game $F_1^N, \dots, F_N^N$. Then up to extraction of a subsequence, the sequence of measures $m_{\bar{X}^N}^N$ converges to a measure $\bar{m} \in \mathcal{P}(Q)$ such that*

$$\int_Q F(y, \bar{m})d\bar{m}(y) = \inf_{m\in\mathcal{P}(Q)} \int_Q F(y, \bar{m})dm(y) \ . \tag{3.6}$$

Remark 3.12. The "mean field equation" (3.6) is equivalent to saying that the support of $\bar{m}$ is contained in the set of minima of $F(y, \bar{m})$. Indeed, if $\mathrm{Spt}(\bar{m}) \subset \mathrm{argmin}_{y\in Q}F(y, \bar{m})$, then clearly $\bar{m}$ satisfies (3.6). Conversely, if (3.6) holds, then choosing $m = \delta_x$ shows that

$$\int_Q F(y, \bar{m})d\bar{m}(y) \leq F(x, \bar{m}) \text{ for any } x \in Q.$$

Therefore, we have

$$\int_Q F(y, \bar{m})d\bar{m}(y) \leq \min_{x\in Q} F(x, \bar{m}),$$

which implies that $\bar{m}$ is supported in $\mathrm{argmin}_{y\in Q}F(y, \bar{m})$.

Proof. Without loss of generality we can assume that the sequence $m^N_{\bar{X}^N}$ converges to some $\bar{m}$. Let us check that $\bar{m}$ satisfies (3.6). Note that, by the definition of a pure Nash equilibrium, the measure $\delta_{\bar{x}^N_i}$ is a minimizer of the problem

$$\inf_{m\in\mathcal{P}(Q)} \int_Q F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{\bar{x}^N_j})dm(y).$$

Since

$$d_1\Big(\frac{1}{N-1}\sum_{j\neq i}\delta_{\bar{x}^N_j}, m^N_{\bar{X}^N}\Big) \leq \frac{2}{N} ,$$

and since F is uniformly continuous, the measure $\delta_{\bar{x}^N_i}$ is also $\varepsilon-$optimal for the problem

$$\inf_{m\in\mathcal{P}(Q)} \int_Q F(y, m^N_{\bar{X}^N})dm(y),$$

as soon as N is sufficiently large, and this is true for all $i = 1, \ldots, N$. By linearity, so is $m^N_{\bar{X}^N}$:

$$\int_Q F(y, m^N_{\bar{X}^N})dm^N_{\bar{X}^N}(y) \leq \inf_{m\in\mathcal{P}(Q)} \int_Q F(y, m^N_{\bar{X}^N})dm(y) + \varepsilon.$$

Letting $N \to +\infty$ gives the result. □

3.4 Limit of the Nash equilibria in mixed strategies

Theorem 3.11 is not completely satisfying because it requires the existence of a pure Nash equilibrium in the $N-$player game, which does not always hold. However a Nash equilibrium in mixed strategies always exists, and we now discuss the corresponding result.

We now assume that the players play the same game $F^N_1, \ldots, F^N_N$ as before, but they are allowed to play in mixed strategies – they minimize over elements of $\mathcal{P}(Q)$ instead of minimizing over elements of Q. If the players play the mixed strategies $\pi_1, \ldots, \pi_N \in \mathcal{P}(Q)$, then the outcome of player i (still denoted, by abuse of notation, F^i_N) is

$$F^N_i(\pi_1, \ldots, \pi_N) = \int_{Q^N} F\Big(x_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\Big)d\pi_1(x_1)\ldots d\pi_N(x_N) . \tag{3.7}$$

Recall that that symmetric Nash equilibria do exist for mixed strategies, unlike for pure strategies.

Theorem 3.13. *Assume that F is Lipschitz continuous. Let $(\bar{\pi}^N, \ldots, \bar{\pi}^N)$ be a symmetric Nash equilibrium in mixed strategies for the game $F^N_1, \ldots, F^N_N$. Then, up to a subsequence, $\bar{\pi}^N$ converges to a measure $\bar{m}$ satisfying (3.6).*

Remark 3.14. In particular the above Theorem proves the existence of a solution to the "mean field equation" (3.6).

Proof. Let $\bar{m}$ be a limit, up to extracting a subsequence, of $\bar{\pi}^N$. Fix $y \in Q$ and consider the map

$$\tilde{F}(y,x) = F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}) : \ Q^{N-1} \to \mathbb{R}.$$

Note that $\tilde{F}$ is $\mathrm{Lip}(F)/(N-1)-$Lipschitz continuous in each coordinate $x_j \in Q$, hence twe have, by the definition of the distance d_1:

$$\Big|\int_{Q^{N-1}} \tilde{F}(y,x)\prod_{j\neq i} d\bar{\pi}^N(x_j) - \int_{Q^{N-1}} \tilde{F}(y,x)\prod_{j\neq i} d\bar{m}(x_j)\Big| \leq \mathrm{Lip}(F) d_1(\bar{\pi}^N, \bar{m}) \qquad \forall y \in Q \ . \tag{3.8}$$

Since $(\bar{\pi}_1, \ldots, \bar{\pi}_N)$ is a Nash equilibrium, inequality (3.8) implies that, for any $\varepsilon > 0$ and if we choose N large enough, we have

$$\int_{Q^N} F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j})\prod_{j\neq i} d\bar{m}(x_j)d\bar{m}(y) \leq \int_{Q^N} F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j})\prod_{j\neq i} d\bar{m}(x_j)dm(y) + \varepsilon \ , \tag{3.9}$$

for any $m \in \mathcal{P}(Q)$. Note also that we have

$$\lim_{N\to+\infty} \int_{Q^{N-1}} F(y, \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j})\prod_{j\neq i} d\bar{m}(x_j) = F(y,\bar{m}) \ , \tag{3.10}$$

where the convergence is uniform with respect to $y \in Q$ thanks to the (Lipschitz) continuity of F. Letting $N \to +\infty$ in both sides of (3.9) gives, in view of (3.10),

$$\int_Q F(y,\bar{m})d\bar{m}(y) \leq \int_Q F(y,\bar{m})dm(y) + \varepsilon \qquad \forall m \in \mathcal{P}(Q) \ ,$$

which finishes the proof, since ε is arbitrary. □

We can also investigate the converse statement: suppose that a measure $\bar{m}$ satisfying the equilibrium condition (3.6) is given. To what extent can it be used in an $N-$player game?

Theorem 3.15. *Let F be as in Theorem 3.13. For any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that, if $N \geq N_0$, the symmetric mixed strategy $(\bar{m}, \cdot, \bar{m})$ is $\varepsilon-$optimal in the $N-$player game with costs (F_i^N) defined by (3.7). Namely, we have*

$$F_i^N(\bar{m}, \ldots, \bar{m}) \leq F_i^N(x_i, (\bar{m})_{j\neq i}) + \varepsilon \qquad \forall x_i \in Q.$$

Proof. Indeed, as explained in the proof of Theorem 3.13, see (3.10). we have

$$\lim_{N\to+\infty} F_i^N(x_i, (\bar{m})_{j\neq i}) = F(x_i, \bar{m})$$

and this limit holds uniformly with respect to $x_i \in Q$. So we can find N_0 such that

$$\sup_{x_i\in Q} \big|F_i^N(x_i, (\bar{m})_{j\neq i}) - F(x_i,\bar{m})\big| \leq \varepsilon/2 \qquad \forall N \geq N_0. \tag{3.11}$$

Then, for any $x_i \in Q$, we have

$$F_i^N(x_i, (\bar{m})_{j\neq i}) \geq F(x_i, \bar{m}) - \varepsilon/2 \geq \int_Q F(y_i, \bar{m}) d\bar{m}(y_i) - \varepsilon/2 \tag{3.12}$$

where the last inequality comes from the mean-field equaiton (3.6) for $\bar{m}$ by using $m = \delta_{x_i}$. Using again (3.11) and (3.12), we finally get

$$F_i^N(x_i, (\bar{m})_{j\neq i}) \geq \int_Q F(y_i, \bar{m}) d\bar{m}(y_i) - \varepsilon/2 \geq F_i^N(\bar{m}, \ldots, \bar{m}) - \varepsilon.$$

□

3.5 A uniqueness result

One obtains the full convergence of the measure $m^N_{\bar{X}^N}$ (or $\bar{\pi}^N$), rather than along a subsequence, if there is a unique measure $\bar{m}$ satisfying the mean-field equation (3.6). This is the case under the following (very strong) assumption:

Proposition 3.16. *Assume that F satisfies*

$$\int_Q (F(y, m_1) - F(y, m_2)) d(m_1 - m_2)(y) > 0 \qquad \forall m_1 \neq m_2 \, . \tag{3.13}$$

Then there is at most one measure satisfying (3.6).

Remark 3.17. Requiring at the same time the continuity of F and the above monotonicity condition seems rather restrictive for applications.

Condition (3.13) is more easily fulfilled for mappings defined on strict subsets of $\mathcal{P}(Q)$. For instance, if Q is a compact subset of $\mathbb{R}^d$ of positive measure and $\mathcal{P}_{ac}(Q)$ is the set of absolutely continuous measures on Q, with respect to the Lebesgue measure, then

$$F(y, m) = \begin{cases} G(m(y)) & \text{if } m \in \mathcal{P}_{ac}(Q) \\ +\infty & \text{otherwise} \end{cases}$$

satisfies (3.13) as soon as $G : \mathbb{R} \to \mathbb{R}$ is continuous and increasing. Here, we denote by $m(y)$ the density of m at y.

If we assume that Q is the closure of a smooth open bounded subset Ω of $\mathbb{R}^d$, another example is given by

$$F(y, m) = \begin{cases} u_m(y) & \text{if } m \in \mathcal{P}_{ac}(Q) \cap L^2(Q) \\ +\infty & \text{otherwise} \end{cases}$$

where u_m is the solution in $H^1(Q)$ of

$$\begin{cases} -\Delta u_m = m & \text{in } \Omega \\ u_m = 0 & \text{on } \partial\Omega \end{cases}$$

Note that in this case the map $y \to F(y, m)$ is continuous.

Proof of Proposition 3.16. Let $\bar{m}_1, \bar{m}_2$ satisfying (3.6). Then

$$\int_Q F(y, \bar{m}_1) d\bar{m}_1(y) \leq \int_Q F(y, \bar{m}_1) d\bar{m}_2(y)$$

and

$$\int_Q F(y, \bar{m}_2) d\bar{m}_2(y) \leq \int_Q F(y, \bar{m}_2) d\bar{m}_1(y) \ .$$

Therefore

$$\int_Q (F(y, \bar{m}_1) - F(y, \bar{m}_2)) d(\bar{m}_1 - \bar{m}_2)(y) \leq 0 \ ,$$

which implies that $\bar{m}_1 = \bar{m}_2$ thanks to assumption (3.13). □

3.6 An example: potential games

We now consider a class of nonatomic games for which the mean-field game equilibria can be found by minimizing a functional. To fix the idea, we assume that $Q \subset \mathbb{R}^d$, and that $F(x, m)$ has the form

$$F(y, m) = \begin{cases} F(m(y)) & \text{if } m \in \mathcal{P}_{ac}(Q) \\ +\infty, & \text{otherwise} \end{cases}$$

where $\mathcal{P}_{ac}(Q)$ is the set of absolutely continuous measures on Q, with respect to the Lebesgue measure, and $m(y)$ is the density of m at $y \in Q$. If $F(x, m)$ can be represented as the derivative of some mapping $\Phi(x, m)$ with respect to the $m-$variable, and if the problem

$$\inf_{m \in \mathcal{P}(Q)} \int_Q \Phi(x, m) dx$$

has a minimum $\bar{m}$, then the first variation tells us that

$$\int_Q \Phi'(x, \bar{m})(dm - d\bar{m}) \geq 0 \qquad \forall m \in \mathcal{P}(Q),$$

so

$$\int_Q F(x, \bar{m}) dm \geq \int_Q F(x, \bar{m}) d\bar{m} \qquad \forall m \in \mathcal{P}(Q) \ ,$$

which shows that $\bar{m}$ is a solution of the mean-field game equation.

For instance let us assume that

$$F(x, m) = \begin{cases} V(x) + G(m(x)) & \text{if } m \in \mathcal{P}_{ac}(Q) \\ +\infty & \text{otherwise} \end{cases}$$

where $V : Q \to \mathbb{R}$ is continuous and $G : (0, +\infty) \to \mathbb{R}$ is continuous, strictly increasing, with $G(0) = 0$ and $G(s) \geq cs$ for some $c > 0$. Then let

$$\Phi(x, m) = V(x)m(x) + H(m(x)) \qquad \text{if } m \text{ is a.c.}$$

where H is a primitive of G with $H(0) = 0$. Note that H is strictly convex with

$$H(s) \geq \frac{c}{2} s^2.$$

Hence the problem

$$\inf_{m \in \mathcal{P}_{ac}(Q)} \int_Q V(x)m(x) + H(m(x))dx$$

has a unique solution $\bar{m} \in L^2(Q)$. Then we have, for any $m \in \mathcal{P}_{ac}(Q)$,

$$\int_Q (V(x) + G(\bar{m}(x)))m(x)dx \geq \int_Q (V(x) + G(\bar{m}(x)))\bar{m}(x)dx\ ,$$

so that $\bar{m}$ satisfies (a slightly modified version of) the mean field equation (3.6). In particular, we have

$$V(x) + G(m(x)) = \min_y V(y) + G(\bar{m}(y)) \text{ for any } x \in Spt(\bar{m}).$$

Let us set $\lambda = \min_y V(y) + G(\bar{m}(y))$. Then

$$\bar{m}(x) = G^{-1}((\lambda - V(x))_+)$$

For instance, if we plug formally $Q = \mathbb{R}^d$, $V(x) = |x|^2/2$ and $G(s) = \log(s)$ into the above equality, we get $m(x) = e^{-|x|^2/2}/(2\pi)^{d/2}$.

3.7 Comments

There is a huge literature on games with a continuum of players, starting from the seminal work by Aumann [14]. Schmeidler [120], and then Mas-Colell [111], introduced a notion of non-cooperative equilibrium in games with a continuum of agents and established several existence results in a much more general framework where the agents have *types*, i.e., personal characteristics; in that set-up, the equilibria are known under the name of Cournot-Nash equilibria. Blanchet and Carlier [19] investigated classes of problems in which such equilibrium is unique and can be fully characterized.

The variational approach described in Section 3.6 presents strong similarities with the potential games of Monderer and Shapley [113].

4 The mean field game system with a non-local coupling

This part is devoted to the mean field game (MFG) system

$$\begin{array}{rl} (i) & -\partial_t u - \Delta u + H(x, Du) = F(x, m) \\ (ii) & \partial_t m - \Delta m - \operatorname{div}(m D_p H(x, Du(t,x))) = 0 \\ (iii) & m(0) = m_0\ ,\ u(T,x) = G(x, m(T)). \end{array} \tag{4.1}$$

The Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is assumed to be convex with respect to the second variable. The two equations in (4.1) are coupled via the functions F and G. For simplicity, we work with the data which are periodic in space: although this situation is completely unrealistic in terms of applications, this assumption simplifies the proofs and avoids the technical discussion on the boundary conditions. Note that we have set the diffusivity to be equal to one, to simplify the notation. We will also consider the case when the diffusivity vanishes, so the system is of the first order.

The MFG system can be interpreted as a Nash equilibrium for a system for nonatomic agents with a cost (or pay-off) depending of the density of the other agents. More precisely, at the initial time $t = 0$ the agents are distributed according to the probability density m_0. We make the strong assumption that the agents also share a common belief on the future behavior of the density of agents $m(t)$, with, of course, $m(0) = m_0$. Each player, if he starts from a position x at time $t = 0$, has to solve a problem of the form

$$\inf_{\alpha} \mathbb{E}\left[\int_0^T (H^*(X_s, \alpha_s) + F(X_s, m(s)))ds + G(X_T, m(T))\right]$$

where H^* is the Legendre transform of H with respect to the last variable, as in (2.26):

$$H(p, x) = \inf_{a\in A} \left[H^*(x, a) + a \cdot p\right], \quad p \in \mathbb{R}^d, \tag{4.2}$$

and X_s is the solution to the SDE

$$dX_s = \alpha_s ds + \sqrt{2} dB_s, \qquad X_0 = x.$$

Here, B_s is a standard $d-$dimensional Brownian motion and the infimum is taken over controls $\alpha : [0, T] \to \mathbb{R}^d$ adapted to the filtration generated by B_s. Note that the final cost $G(X_T, m(T))$ depends not only on the final position but also on the distribution of the other players at the final time T, and that the running cost $F(X_s, m(s))$ depends on the position X_s and $m(s)$ but not directly on the control α_s.

As it is standard in the control theory, it is convenient to introduce the value function $u(t, x)$ for this problem:

$$u(t, x) := \inf_{\alpha} \mathbb{E}\left[\int_t^T (H^*(X_s, \alpha_s) + F(X_s, m_s))ds + G(X_T, m_T))\right]$$

where

$$dX_s = \alpha_s ds + \sqrt{2} dB_s,\ X_t = x.$$

As we have discussed, if m_t is known, then u is a classical solution to the Hamilton-Jacobi equation (4.1)-(i) with the terminal condition $u(T, x) = G(x, m_T)$. Moreover, the optimal feedback of each agent is given by

$$\alpha^*(t, x) := -D_p H(x, Du(t, x)).$$

Hence, the best policy for each individual agent at position x at time t, is to play $\alpha^*(t, x)$. Then, the actual density $\tilde{m}(t)$ of agents would evolve according to the Fokker-Planck equation (4.1)-(ii), with the initial condition $\tilde{m}(0) = m_0$. We say that the pair (u, m) is a Nash equilibrium of the game if the pair (u, m) satisfies the MFG system (4.1). This agrees with our discussion in the previous section.

We discuss here several regimes for the MFG system: first, the uniformly parabolic case, for which existence of a classical solution for the system is expected to hold. When there is no diffusion, one has to introduce a suitable notion of a weak solution. We will also have to consider various smoothing properties of the couplings F and G, depending on whether the couplings are regularizing or not. This is what leads us to separate the "more regularizing" non-local couplings from "not so much regularizing" local couplings.

4.1 The existence theorem

Let us start with the second order mean field games with a nonlocal coupling:

$$\begin{array}{rl} (i) & -\partial_t u - \Delta u + H(x, Du) = F(x, m) \qquad \text{in } (0,T)\times\mathbb{T}^d, \\ (ii) & \partial_t m - \Delta m - \operatorname{div}\left(m\ D_pH(x, Du(t,x))\right) = 0 \qquad \text{in } (0,T)\times\mathbb{T}^d, \\ (iii) & m(0) = m_0\ ,\ u(T,x) = G(x, m(T)) \qquad \text{in } \mathbb{T}^d. \end{array} \tag{4.3}$$

Our aim is to prove the existence of classical solutions for this system and give some interpretation in terms of a game with finitely many players.

Let us describe various assumptions used throughout the section. Our main hypothesis is that F and G are regularizing on the set of probability measures on $\mathbb{T}^d$ in the following sense. Let $\mathcal{P}(\mathbb{T}^d)$ be the set of such Borel probability measures on $\mathbb{T}^d$ endowed with the Kantorovitch-Rubinstein distance:

$$d_1(\mu,\nu) = \sup\left\{ \int_{\mathbb{T}^d} \phi(x)(\mu-\nu)(dx) \text{ s.t } \phi : \mathbb{T}^d \to \mathbb{R} \text{ is 1-Lipschitz continuous}\right\}. \tag{4.4}$$

Recall that the distance metricizes the weak-* topology on $\mathcal{P}(\mathbb{T}^d)$ and that $\mathcal{P}(\mathbb{T}^d)$ is a compact space.

Here are our main assumptions on F, G and m_0:
(i) The functions $F(x,m)$ and $G(x,m)$ are Lipschitz continuous in $\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$,
(ii) Uniform regularity of F and G in space: $F(\cdot,m)$ and $G(\cdot,m)$ are bounded in $C^{1+\beta}(\mathbb{T}^d)$ and $C^{2+\beta}(\mathbb{T}^d)$ (for some $\beta\in(0,1)$) uniformly with respect to $m\in\mathcal{P}(\mathbb{T}^d)$.
(iii) The Hamiltonian $H:\mathbb{T}^d\times\mathbb{R}^d\to\mathbb{R}$ is locally Lipschitz continuous, D_pH exists and is continuous on $\mathbb{T}^d\times\mathbb{R}^d$, and H satisfies the growth condition

$$\langle D_xH(x,p), p\rangle \geq C_0(1+|p|^2) \tag{4.5}$$

for some constant $C_0>0$.
(iv) The probability measure m_0 is absolutely continuous with respect to the Lebesgue measure, and has a $C^{2+\beta}$ continuous density, still denoted m_0.

Let us comment on the Lipschitz continuity in m assumption. For example, if we fix a Lipschitz function f and take

$$F(m) = \int_{\mathbb{T}^d} f(x)dm,$$

then

$$|F(m_1) - F(m_2)| \leq \|f\|_{Lip} d_1(m_1, m_2),$$

thus $F(m)$ is Lipschitz continuous. On the other hand, if we take a function $g:\mathbb{R}\to\mathbb{R}$ and define $F(x,m) = g(m(x))$ for measures m that are absolutely continuous with respect to the

Lebesgue measure, then $F(x, m)$ is not Lipschitz continuous in the d_1-metric on $\mathcal{P}(\mathbb{T}^d)$, no matter how nice g is. This is why this assumption implies that coupling is non-local.

A pair (u, m) is a classical solution to (4.3) if $u, m : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ are continuous, of class C^2 in space and C^1 in time and (u, m) satisfies (4.3) in the classical sense. The main result of this section is the following:

Theorem 4.1. *Under the above assumptions, there is at least one classical solution to (4.3).*

The proof is relatively easy and relies on the basic estimates for Hamilton-Jacobi equations and on some remarks on the Fokker-Planck equation (4.3-(ii)). We give the details below.

4.2 On the Fokker-Planck equation

Let $b : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ be a given vector field. Our aim is to analyse the Fokker-Planck equation

$$\begin{cases} \partial_t m - \Delta m - \operatorname{div}(mb) = 0 & \text{in } \mathbb{T}^d \times (0, T), \\ m(0, x) = m_0(x), \end{cases} \tag{4.6}$$

as an evolution equation in the space of probability measures. We assume here that the vector field $b : \mathbb{T}^d \times [0, T] \to \mathbb{R}^d$ is continuous in time and Hölder continuous in space.

Definition 4.2. *We say that $m \in L^1(\mathbb{T}^d \times [0, T])$ is a weak solution to (4.6) if for any test function $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, T))$, we have*

$$\begin{aligned} &\int_{\mathbb{T}^d} \phi(x, 0) dm_0(x) - \int_{\mathbb{T}^d} \phi(x, t) dm(t)(x) \\ &+ \int_0^T \int_{\mathbb{T}^d} \left(\partial_t \varphi(t, x) + \Delta \varphi(t, x) - \langle D\varphi(t, x), b(t, x) \rangle\right) dm(t)(x) = 0. \end{aligned}$$

In order to analyze some particular solutions of (4.6), it is convenient to introduce the following stochastic differential equation (SDE)

$$\begin{cases} dX_t = -b(X_t, t)dt + \sqrt{2} dB_t, & t \in [0, T] \\ X_0 = Z_0 \end{cases} \tag{4.7}$$

where the initial condition $Z_0 \in L^1(\Omega)$ is possibly random and independent of B_t. Under the above assumptions on b, there is a unique solution to (4.7). This solution is closely related to equation (4.6):

Lemma 4.1. *If $\mathcal{L}(Z_0) = m_0$, then $m(t) := \mathcal{L}(X_t)$ a weak solution of (4.6).*

Proof. This is a straightforward consequence of the Itô formula: if $\varphi(t, x)$ is smooth with compact support, then

$$\begin{aligned} \varphi(X_t, t) = \varphi(Z_0, 0) &+ \int_0^t \left[\partial_s \varphi(X_s, s) - \langle D\varphi(X_s, s), b(X_s, s) \rangle + \Delta \varphi(X_s, s)\right] ds \\ &+ \int_0^t \langle D\varphi(X_s, s), dB_s \rangle \,. \end{aligned}$$

Taking the expectation on both sides (with respect to the Brownian motion and the randomness in the initial condition) gives

$$\mathbb{E}\left[\varphi(X_t,t)\right] = \mathbb{E}\Big[\varphi(Z_0,0) + \int_0^t \left[\varphi_t(X_s,s) - \langle D\varphi(X_s,s), b(X_s,s)\rangle + \Delta\varphi(X_s,s)\right] ds\Big] .$$

So by definition of $m(t)$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(t,x)dm(t)(x) &= \int_{\mathbb{R}^d} \varphi(x,0)dm_0(x) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \left[\varphi_t(x,s) - \langle D\varphi(x,s), b(x,s)\rangle + \Delta\varphi(x,s)\right] dm(s)(x)ds,\end{aligned}$$

thus m is a weak solution to (4.6). □

The interpretation of the solution of the continuity equation as the law of the corresponding solution of the SDE allows us to get a Hölder regularity estimate on $m(t)$ in $\mathcal{P}(\mathbb{T}^d)$.

Lemma 4.2. *There is a constant $c_0 = c_0(T)$, independent of $\nu \in (0,1]$, such that*

$$d_1(m(t), m(s)) \leq c_0(1 + \|b\|_\infty)|t-s|^{1/2} \qquad \forall s,t \in [0,T] .$$

Proof. We write

$$\begin{aligned}&d_1(m(t),m(s)) = \sup\Big\{ \int_{\mathbb{T}^d} \phi(x)(m(t)-m(s))(dx) \text{ s.t } \phi \text{ is 1-Lipschitz continuous}\Big\} \\ &\leq \sup\Big\{\mathbb{E}\left[\phi(X_t) - \phi(X_s)\right] \text{ s.t } \phi \text{ is 1-Lipschitz continuous}\Big\} \leq \mathbb{E}\left[|X_t - X_s|\right].\end{aligned}$$

Moreover, if, for instance, $s < t$ we have

$$\mathbb{E}\left[|X_t - X_s|\right] \leq \mathbb{E}\left[\int_s^t |b(X_\tau,\tau)|\, d\tau + \sqrt{2}\,|B_t - B_s|\right] \leq \|b\|_\infty(t-s) + \sqrt{2\nu(t-s)}.$$

This finishes the proof. □

4.3 Proof of the existence Theorem

We are now ready to prove Theorem 4.1. For a large constant C_1 to be chosen below, let $\mathcal{C}$ be the set of maps $\mu \in C^0([0,T], \mathcal{P}(\mathbb{T}^d))$ such that

$$\sup_{s\neq t} \frac{d_1(\mu(s),\mu(t))}{|t-s|^{1/2}} \leq C_1. \tag{4.8}$$

Then $\mathcal{C}$ is a convex closed subset of $C^0([0,T], \mathcal{P}(\mathbb{T}^d))$. It is actually compact thanks to Ascoli's Theorem and the compactness of the set $\mathcal{P}(\mathbb{T}^d)$.

The proof is based on a fixed point theorem. To any $\mu \in \mathcal{C}$, we associate $m = \Psi(\mu) \in \mathcal{C}$ as follows. Let u be the unique solution to the terminal problem

$$\begin{aligned}&-\partial_t u - \Delta u + H(x, Du) = F(x, \mu(t)) \quad \text{in } (0,T)\times\mathbb{T}^d, \\ &u(x,T) = G(x, \mu(T)) \qquad \text{in } \mathbb{T}^d.\end{aligned} \tag{4.9}$$

Then we define $m = \Psi(\mu)$ as the solution of the initial value problem for the Fokker-Planck equation

$$\begin{aligned} &\partial_t m - \Delta m - \operatorname{div}(m D_p H(x, Du)) = 0 \quad \text{in } (0,T)\times\mathbb{T}^d, \\ &m(0,x) = m_0(x) \qquad \text{in } \mathbb{T}^d. \end{aligned} \tag{4.10}$$

Let us check that Ψ is a well-defined and continuous map $\mathcal{C} \to \mathcal{C}$. Let us set

$$\tilde{H}(t,x,p) = H(x,p) - F(x,\mu(t)).$$

The theory of the viscous Hamilton-Jacobi equations shows that under assumption (4.5) equation (4.9) has a unique classical solution u. Moreover, we have an estimate

$$\|u\|_{C^{2+\alpha,1+\alpha/2}} \le C, \tag{4.11}$$

where $\alpha > 0$ and $C > 0$ do not depend on μ, because of the a priori bounds on F we have assumed. The constant C may depend on T though.

Next we turn to the Fokker-Planck equation (4.10), that we write in the form

$$\partial_t m - \Delta m - \langle Dm, D_p H(x,Du)\rangle - m\ \operatorname{div}[D_p H(x,Du)] = 0\ .$$

Since $u \in C^{2+\alpha,1+\alpha/2}$, the maps $(t,x) \to D_p H(x,Du)$ and $(t,x) \to \operatorname{div} D_p H(x,Du)$ belong to C^α, so that this advection-diffusion equation is uniquely solvable and the solution m belongs to $C^{2+\alpha,1+\alpha/2}$. Moreover, from Lemma 4.2, we have the following estimate on m:

$$d_1(m(t),m(s)) \le c_0(1 + \|D_p H(\cdot,Du)\|_\infty)|t-s|^{1/2} \qquad \forall s,t \in [0,T],$$

where $\|D_p H(\cdot,Du)\|_\infty$ is bounded by a constant C_2 independent of μ, because Du is uniformly bounded due to (4.11). Thus, if we choose C_1 in (4.8) sufficiently large, then m belongs to $\mathcal{C}$, and the mapping $\Psi : \mu \to m = \Psi(\mu)$ is well-defined from $\mathcal{C}$ into itself.

Let us check that Ψ is a continuous map $\mathcal{C} \to \mathcal{C}$. Let us assume that $\mu_n \to \mu$ in $\mathcal{C}$, and let (u_n, m_n) and (u,m) be the corresponding solutions to (4.9)-(4.10). Note that

$$F(x,\mu_n(t)) \to (x,\mu(t)) \text{ and } G(x,\mu_n(T) \to G(x,\mu(T),$$

both uniformly, over $\mathbb{T}^d \times [0,T]$ and $\mathbb{T}^d$, respectively, thanks to our continuity assumptions on F and G. Moreover, as the right side of the Hamilton-Jacobi equation for u_n is bounded in $C^{1+\alpha,1+\alpha/2}$, the functions u_n are uniformly bounded in $C^{2+\alpha,1+\alpha/2}$ so that u_n converges in $C^{2,1}$ to the unique solution u of the Hamilton-Jacobi equation with the right side $F(x,\mu)$. The measures m_n are then solutions of a linear Fokker-Planck equation with uniformly Hölder continuous coefficients, which provides uniform $C^{2+\alpha,1+\alpha/2}$ estimates on m_n. Thus, m_n converge in turn, also in $C^{2,1}$, to the unique solution m of the Fokker-Planck equation associated to $D_p H(x,Du)$. The convergence is then easily proved to be also in $C^0([0,T],\mathcal{P}(\mathbb{T}^d))$. Now, the Schauder fixed point theorem implies that the continuous map $\mu \to m = \Psi(\mu)$ has a fixed point in $\mathcal{C}$: this fixed point (and the corresponding u) is a solution to (4.3).

4.4 Uniqueness of the solution

Let us assume that, besides assumptions given at the beginning of the section, the following conditions hold:

$$\int_{\mathbb{T}^d} (F(x,m_1) - F(x,m_2))d(m_1 - m_2)(x) \geq 0 \qquad \forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^d) \tag{4.12}$$

and

$$\int_{\mathbb{T}^d} (G(x,m_1) - G(x,m_2))d(m_1 - m_2)(x) \geq 0 \qquad \forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^d)\ . \tag{4.13}$$

We also assume that H is uniformly convex with respect to the last variable:

$$\frac{1}{C} I_d \leq D^2_{pp} H(x,p) \leq C I_d, \tag{4.14}$$

with some $C > 0$.

Theorem 4.3. *Under the above conditions, there is a unique classical solution to the mean field equation (4.3).*

Proof. Let (u_1, m_1) and (u_2, m_2) be two classical solutions of (4.3), and set

$$\bar{u} = u_1 - u_2, \quad \bar{m} = m_1 - m_2,$$

then

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{T}^d} \bar{u}\bar{m} dx = \int_{\mathbb{T}^d} [(\partial_t \bar{u})\bar{m} + \bar{u}(\partial_t \bar{m})]dx \\ &= \int_{\mathbb{T}^d} (-\Delta \bar{u} + H(x,Du_1) - H(x,Du_2) - F(x,m_1) + F(x,m_2))\bar{m} dx \\ &+ \int_{\mathbb{T}^d} \bar{u}(\Delta \bar{m} + \operatorname{div}(m_1 D_p H(x,Du_1)) - \operatorname{div}(m_2 D_p H(x,Du_2)))dx. \end{aligned} \tag{4.15}$$

Note that

$$\int_{\mathbb{T}^d} -(\Delta \bar{u})\bar{m} + \bar{u}(\Delta \bar{m}) dx = 0,$$

and, from the monotonicity condition on F, we have

$$\int_{\mathbb{T}^d} (-F(x,m_1) + F(x,m_2))\bar{m} dx = \int_{\mathbb{T}^d} (-F(x,m_1) + F(x,m_2))(m_1 - m_2) dx \leq 0.$$

We now rewrite the remaining terms in (4.15) in the following way:

$$\begin{aligned} R := &\int_{\mathbb{T}^d} [(H(x,Du_1) - H(x,Du_2))\bar{m} - \langle D\bar{u}, m_1 D_p H(x,Du_1) - m_2 D_p H(x,Du_2)\rangle]dx \\ &= -\int_{\mathbb{T}^d} m_1 \left[H(x,Du_2) - H(x,Du_1) - \langle D_p H(x,Du_1), Du_2 - Du_1\rangle\right] dx \\ &- \int_{\mathbb{T}^d} m_2 \left[H(x,Du_1) - H(x,Du_2) - \langle D_p H(x,Du_2), Du_1 - Du_2\rangle\right] dx. \end{aligned}$$

The uniform convexity assumption (4.14) on H implies that

$$R \leq -\int_{\mathbb{T}^d} \frac{(m_1+m_2)}{2C}|Du_1 - Du_2|^2 dx \leq 0.$$

Putting the estimates together we get

$$\frac{d}{dt}\int_{\mathbb{T}^d} \bar{u}\bar{m}dx \leq 0. \tag{4.16}$$

We integrate this inequality on the time interval $[0,T]$ to obtain

$$\int_{\mathbb{T}^d} \bar{u}(T)\bar{m}(T)dx \leq \int_{\mathbb{T}^d} \bar{u}(0)\bar{m}(0)dx - \int_0^T \int_{\mathbb{T}^d} \frac{(m_1+m_2)}{2C}|Du_1 - Du_2|^2 dx. \tag{4.17}$$

Note that $\bar{m}(0) = 0$ while, as $\bar{u}(T) = G(x, m_1(T)) - G(x, m_2(T))$, we have

$$\int_{\mathbb{T}^d} \bar{u}(T)\bar{m}(T)dx = \int_{\mathbb{T}^d} (G(x,m_1(T)) - G(x,m_2(T)))(m_1(T) - m_2(T))dx \geq 0$$

thanks to the monotonicity assumption on G. Now, (4.17) implies that

$$\int_{\mathbb{T}^d} \bar{u}(T)\bar{m}(T)dx = 0,$$

but also that

$$Du_1 = Du_2 \text{ in } \{m_1 > 0\}\bigcup\{m_2 > 0\}.$$

As a consequence, m_2 actually solves the same equation as m_1, with the same drift

$$D_pH(x, Du_1) = D_pH(x, Du_2),$$

hence $m_1 = m_2$. Then, in turn, implies that u_1 and u_2 solve the same Hamilton-Jacobi equation, so that $u_1 = u_2$. □

4.5 An application to games with finitely many players

Before starting the discussion of games with a large number of players, let us fix a solution (u,m) of the mean field system (4.3) and investigate the optimal strategy of a generic player who considers the density m "of the other players" as given. He faces the following minimization problem

$$\inf_\alpha \mathcal{J}(\alpha) \qquad \text{where} \qquad \mathcal{J}(\alpha) = \mathbb{E}\left[\int_0^T L(X_s,\alpha_s) + F\left(X_s, m(s)\right)\, ds + G\left(X_T, m(T)\right)\right].$$

In the above formula, L is a kind of Legendre transform of H with respect to the last variable:

$$L(x,\xi) := \sup_{p\in\mathbb{R}^d}[-\langle p,\xi\rangle - H(x,p)].$$

The process X_t is given by

$$X_t = X_0 + \int_0^t \alpha_s ds + \sqrt{2} B_s,$$

with X_0 a fixed random initial condition with the law m_0, independent of B_t, and the control α is adapted to the filtration $\mathcal{F}_t$ of the $d-$dimensional Brownian motion B_t. We claim that the feedback strategy $\alpha^*(t,x) := -D_p H(x, Du(t,x)$ is optimal for this optimal stochastic control problem.

Lemma 4.3. *Let $\bar{X}_t$ be the solution of the stochastic differential equation*

$$\begin{cases} d\bar{X}_t = \alpha^*(t, \bar{X}_t)dt + \sqrt{2} dB_t \\ \bar{X}_0 = X_0 \end{cases}$$

and set $\bar{\alpha}(t) = \alpha^*(t, X_t)$. Then

$$\inf_{\alpha} \mathcal{J}(\alpha) = \mathcal{J}(\bar{\alpha}) = \int_{\mathbb{R}^N} u(0,x)\ dm_0(x)\ .$$

Proof. This kind of result is known as a verification Theorem: one has a good candidate for an optimal control, and one checks, using the equation satisfied by the value function u, that this is indeed the minimum. Let α be an adapted control. We have, by the Itô formula,

$$\begin{aligned} &\mathbb{E}[G(X_T, m(T))] = \mathbb{E}[u(X_T, T)] \\ &\quad = \mathbb{E}\left[u(0, X_0) + \int_0^T (\partial_t u(s, X_s) + \langle \alpha_s, Du(s, X_s)\rangle + \Delta u(s, X_s))\ ds\right] \\ &\quad = \mathbb{E}\left[u(0, X_0) + \int_0^T (H(X_s, Du(s, X_s)) + \langle \alpha_s, Du(s, X_s)\rangle - F(X_s, m(s)))\ ds\right], \end{aligned}$$

where we have used the equation satisfied by u in the last equality. Thus, by definition of L,

$$\mathbb{E}[G(X_T, m(T))] \geq \mathbb{E}\left[u(0, X_0) + \int_0^T (-L(X_s, \alpha_s) - F(X_s, m(s)))\ ds\right].$$

This shows that

$$\mathbb{E}\left[u(0, X_0)\right] \leq \mathbb{E}\left[\int_0^T (L(X_s, \alpha_s) + F(X_s, m(s)))\ ds + G(X_T, m(T))\right] = J(\alpha)$$

for any adapted control α. If we replace α by $\bar{\alpha}$ in the above computations, then, since

$$\begin{aligned} H(\bar{X}_s, Du(s, \bar{X}_s)) + \langle \bar{\alpha}_s, Du(s, \bar{X}_s)\rangle &= H(\bar{X}_s, Du(s, \bar{X}_s)) + \langle \alpha^*(\bar{X}_s, Du(s, \bar{X}_s))\rangle \\ &= -L(\bar{X}_s, \alpha^*(\bar{X}_s, Du(s, \bar{X}_s))) = -L(\bar{X}_s, \bar{\alpha}_s)) \end{aligned}$$

all the above inequalities become equalities, so $\mathbb{E}\left[u(X_0, 0)\right] = \mathcal{J}(\bar{\alpha})$. □

We now consider a differential game with N players which is a finite number of players approximation of the mean field game. In this game, player $i = 1, \ldots, N$, is controlling through his control α^i a dynamics of the form

$$dX_t^i = \alpha_t^i dt + \sqrt{2} dB_t^i. \tag{4.18}$$

The initial conditions X_0^i for this system are also random and all have the law m_0. We assume that all X_0^i and all the Brownian motions B_t^i, $i = 1, \ldots, N$, are independent. Player i can choose his control α^i adapted to the filtration $\mathcal{F}_t = \sigma\{X_0^j,\ B_s^j,\ s \leq t,\ j = 1, \ldots, N\}$ – players "know about each other". His payoff is then given by

$$\mathcal{J}_i^N(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\left[\int_0^T L(X_s^i, \alpha_s^i) + F(X_s^i, m_{X_s}^{N,i})ds + G(X_T^i, m_{X_T}^{N,i})\right],$$

where

$$m_{X_s}^{N,i} := \frac{1}{N-1}\sum_{j \neq i} \delta_{X_s^j}$$

is the empirical distribution of the players X^j, where $j \neq i$. Our aim is to explain that the strategy given by the mean field game is almost optimal for this problem. More precisely, let (u, m) be a classical solution to the MFG system (4.3) and let us define the feedback

$$\alpha^*(t, x) := -D_p H(x, Du(t, x)).$$

With the closed loop strategy α^* one can associate the open-loop control $\bar{\alpha}^i$ obtained by solving the SDE

$$d\bar{X}_t^i = \alpha^*(t, \bar{X}_t^i)dt + \sqrt{2}dB_t^i \tag{4.19}$$

with random initial condition X_0^i and setting $\bar{\alpha}_t^i = \alpha^*(t, \bar{X}_t^i)$. Note that this control is just adapted to the filtration $\mathcal{F}_t^i = \sigma(X_0^i,\ B_s^i,\ s \leq t\}$, and not to the full filtration $\mathcal{F}_t$ defined above – you do not need the precise information about the other players.

Theorem 4.4. *Assume that F and G are Lipschitz continuous in $\mathbb{T}^d \times P(\mathbb{T}^d)$. Then there exists a constant $C > 0$ such that the symmetric strategy $(\bar{\alpha}^1, \ldots, \bar{\alpha}^N)$ is an $\varepsilon-$Nash equilibrium in the game $\mathcal{J}_1^N, \ldots, \mathcal{J}_N^N$ for $\varepsilon := CN^{-1/(d+4)}$: namely*

$$\mathcal{J}_i^N(\bar{\alpha}^1, \ldots, \bar{\alpha}^N) \leq \mathcal{J}_i^N((\bar{\alpha}^j)_{j \neq i}, \alpha^i) + CN^{-1/(d+4)}$$

for any control α^i adapted to the filtration $(\mathcal{F}_t)$ and any $i \in \{1, \ldots, N\}$.

The Lipschitz continuity assumptions on F and G allow to quantify the error. If F and G are just continuous, one can only say that, for any $\varepsilon > 0$, there exists N_0 such that the symmetric strategy $(\bar{\alpha}^1, \ldots, \bar{\alpha}^N)$ is an $\varepsilon-$Nash equilibrium in the game $\mathcal{J}_1^N, \ldots, \mathcal{J}_N^N$ for any $N \geq N_0$.

Before starting the proof, we need the following result on product measures due to Horowitz and Karandikar (see for instance Rashev and Rüschendorf [118], Theorem 10.2.1).

Lemma 4.4. *Assume that Z_i are i.i.d. random variables with a law μ. Then there is a constant C, depending only on d, such that*

$$\mathbb{E}[d_1(m_Z^N, \mu)] \leq CN^{-1/(d+4)}, \qquad \text{where } m_Z^N = \sum_{i=1}^N \delta_{Z_i}.$$

Proof of Theorem 4.4. Fix $\varepsilon > 0$. Since the problem is symmetrical, it is enough to show that

$$\mathcal{J}_1^N(\bar{\alpha}^1, \dots, \bar{\alpha}^N) \leq \mathcal{J}_1^N((\bar{\alpha}^j)_{j\neq 1}, \alpha) + \varepsilon \tag{4.20}$$

for any control α, as soon as N is large enough. Recall that $\bar{X}_t^j$ is the solution of the stochastic differential equation (4.19) with the initial condition X_0^j. We note that $\bar{X}_t^j$ are independent and identically distributed with the law $m(t)$ – see Lemma 4.1. Therefore, using Lemma 4.4, we have for any $t \in [0, T]$,

$$\mathbb{E}\left[d_1(m^{N,i}_{\bar{X}_t}, m(t))\right] \leq CN^{-1/(d+4)}.$$

By Lipschitz continuity of F and G with respect to the variable m, we have therefore:

$$\mathbb{E}\left[\int_0^T \sup_{x\in\mathbb{T}^d} |F(x, m^{N,1}_{\bar{X}_t}) - F(x, m(t))|dt\right] + \mathbb{E}\left[\sup_{x\in\mathbb{T}^d} |G(x, m^{N,1}_{\bar{X}_T}) - G(x, m(T))|\right] \leq CN^{-1/(d+4)}.$$

Let now α^1 be a control adapted to the filtration $\mathcal{F}_t$ and X_t^1 be the solution to

$$dX_t^1 = \alpha_t^1 dt + \sqrt{2} dB_t^1$$

with a random initial condition X_0^1. We have

$$\begin{aligned}
\mathcal{J}_1^N((\bar{\alpha}^j)_{j\neq 2}, \alpha^1) &= \mathbb{E}\Big[\int_0^T (L(X_s^1, \alpha_s^1) + F(X_s^1, m^{N,i}_{\bar{X}_s}))\ ds + G(X_T^1, m^{N,i}_{\bar{X}_T})\Big] \\
&\geq \mathbb{E}\Big[\int_0^T (L(X_s^1, \alpha_s^1) + F\left(X_s^1, m(s)\right))\ ds + G\left(X_T^1, m(T)\right)\Big] - CN^{-1/(d+4)} \\
&\geq \mathcal{J}_1^N((\bar{\alpha}^j)_{j\neq 1}, \bar{\alpha}^1) - CN^{-1/(d+4)}.
\end{aligned}$$

The last inequality comes from the optimality of $\bar{\alpha}$ in Lemma 4.3. This proves the result. □

Remark 4.5. Although sufficient in our context, the estimate

$$\sup_{t\in[0,T]} \mathbb{E}\left[d_1(m^{N,i}_{\bar{X}_t}, m(t))dt\right] \leq CN^{-1/(d+4)}.$$

is a very rough one. One can actually prove that

$$\mathbb{E}\left[\sup_{t\in[0,T]} d_1(m^{N,i}_{\bar{X}_t}, m(t))dt\right] \leq CN^{-1/(d+8)}.$$

See [118], Theorem 10.2.7.

4.6 Extensions

Several other classes of MFG systems have be studied in the literature. We discuss only a few of them, since the number of models has grown exponentially in the last years.

4.6.1 The ergodic MFG system

One may be interested in the large time average of the MFG system (4.3) as the horizon T tends to infinity. It turns out that the limit system takes the following form:

$$\left\{\begin{array}{ll} (i) & \lambda - \Delta u + H(x, Du) = F(x, m) \qquad \text{in } \mathbb{T}^d \\ (ii) & -\Delta m - \operatorname{div}(m\ D_pH(x, Du(x))) = 0 \qquad \text{in } \mathbb{T}^d \end{array}\right. \tag{4.21}$$

Here the unknown are now (λ, u, m), where $\lambda \in \mathbb{R}$ is the so-called ergodic constant. The interpretation of the system is the following: each player wants to minimize his ergodic cost

$$\mathcal{J}(x, \alpha) := \limsup_{T \to +\infty} \inf_{\alpha} \mathbb{E}\left[\frac{1}{T}\int_0^T [H^*(X_t, -\alpha_t) + F(X_t, m(t))]dt\right]$$

where X_t in the solution to

$$\left\{\begin{array}{l} dX_t = \alpha_t dt + \sqrt{2}dB_t \\ X_0 = x \end{array}\right.$$

It turns out that, if (λ, u, m) is a classical solution to (4.21), then the optimal strategy of each tiny player is given by the feedback

$$\alpha^*(t, x) := -D_pH(x, Du(x))$$

and, if $\bar{\alpha}$ is the solution to

$$\left\{\begin{array}{l} dX_t = \alpha^*(t, X_t)dt + \sqrt{2}dB_t \\ X_0 = x \end{array}\right. \tag{4.22}$$

and if we set $\bar{\alpha}_t := \alpha^*(t, X_t)$, then $\mathcal{J}(x, \bar{\alpha}) = \lambda$ is independent of the initial position. Finally, m is the invariant measure associated with the SDE (4.22).

4.6.2 The infinite horizon problem

Another natural model pops up when each player aims at minimizing a infinite horizon cost:

$$\mathcal{J}(x, \alpha) = \inf_{\alpha} \mathbb{E}\left[\int_0^{+\infty} e^{-rt}\left(H^*(X_t, -\alpha_t) + F(X_t, m(t))\right) dt\right]$$

where $r > 0$ is a fixed discount rate. Note that there is no reason for the equilibrium for been given by the initial repartition of the players. This implies that the infinite horizon MFG system is *not* stationary. It is actually system of evolution equations in infinite horizon, given by:

$$\left\{\begin{array}{ll} (i) & -\partial_t u + ru - \Delta u + H(x, Du) = F(x, m(t)) \qquad \text{in } (0, +\infty) \times \mathbb{T}^d \\ (ii) & \partial_t m - \Delta m - \operatorname{div}(m\ D_pH(x, Du(t, x))) = 0 \qquad \text{in } (0, +\infty) \times \mathbb{T}^d \\ (iii) & m(0) = m_0 \text{ in } \mathbb{T}^d, \qquad u \text{ bounded} \end{array}\right.$$

4.6.3 General diffusion

The above results can be extended in several directions. For instance, the diffusion need not be a Brownian motion, and, for the existence part, the system need not be in "separated form". Namely, the existence result (i.e., Theorem 4.1) still holds (with almost the same proof) for the system

$$\left\{\begin{array}{cll} (i) & -\partial_t u - a_{ij}\partial^2_{ij} u + H(x, Du, m(t)) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ (ii) & \partial_t m - \partial_{ij}(a_{ij} m) - \operatorname{div}\left(m\ D_p H(x, Du(t,x), m(t))\right) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ (iii) & m(0) = m_0\ ,\ u(x,T) = G(x, m(T)) & \text{in } \mathbb{T}^d \end{array}\right.$$

where we sum over repeated indices (i.e., $a_{ij}\partial^2_{ij}u := \sum_{i,j=1}^d a_{ij}\partial^2_{ij}u$). In the above expression, we assume that there exists $c_0 > 0$ such that

$$(c_0)^{-1} I_d \leq (a_{i,j}(t,x) \leq c_0 I_d \qquad \forall (t,x) \in [0,T]\times\mathbb{T}^d,$$

that (a_{ij}) is continuous and uniformly Lipschitz continuous with respect to the space variable. We also assume that the Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \times P(\mathbb{T}^d) \to \mathbb{R}$ is locally Lipschitz continuous, $D_p H$ exists and is continuous on $\mathbb{T}^d \times \mathbb{R}^d \times P(\mathbb{T}^d)$, and H satisfies the growth condition (4.5) uniformly with respect to m. We finally suppose that $H(\cdot, p, m)$ is bounded in $C^{1+\beta}(\mathbb{T}^d)$ (for some $\beta \in (0,1)$) locally uniformly with respect to $(p,m) \in \mathbb{R}^d \times P(\mathbb{T}^d)$. The assumptions on G are the same as before.

4.6.4 Neumann boundary conditions

When the small players have to control a process in a domain $\Omega \subset \mathbb{R}^d$ with reflexion on the boundary of Ω, the MFG system takes the form:

$$\left\{\begin{array}{cll} (i) & -\partial_t u - \Delta u + H(x, Du) = F(x,m) & \text{in } (0,T)\times\Omega \\ (ii) & \partial_t m - \Delta m - \operatorname{div}\left(m\ D_p H(x, Du(t,x))\right) = 0 & \text{in } (0,T)\times\Omega \\ (iii) & D_\nu u(t,x) = 0, \qquad D_\nu m(t,x) + \langle D_p H(x, Du(t,x)), \nu\rangle = 0 & \text{in } (0,T)\times\partial\Omega \\ (iv) & m(0) = m_0\ ,\ u(T,x) = G(x, m(T)) & \text{in } \mathbb{T}^d \end{array}\right.$$

where ν is the unit outward normal to Ω.

4.6.5 MFG systems with several populations

We now assume that the system consists in several populations (say, to fix the ideas, I populations). Then the system takes the form

$$\left\{\begin{array}{cll} (i) & -\partial_t u_i - \Delta u_i + H_i(x, Du_i) = F_i(x, m(t)) & \text{in } (0,T)\times\mathbb{T}^d \\ (ii) & \partial_t m_i - \Delta m_i - \operatorname{div}\left(m_i\ D_p H_i(x, Du_i(t,x))\right) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ (iii) & m_i(0) = m_{i,0}\ ,\ u_i(T,x) = G_i(x, m(T)) & \text{in } \mathbb{T}^d \end{array}\right.$$

where $i = 1, \dots, I$, u_i denotes the value function of each player in population i and $m = (m_1, \dots, m_I)$ denotes the collection of densities m_i of the population i. The coupling functions F_i and G_i depend on the all the densities. Existence of solutions can be proved by fixed point arguments as in Theorem 4.1. Uniqueness, however, is a difficult issue.

4.7 Comments

Existence: Existence of solutions for the MFG system can be achieved either by Banach fixed point Theorem (as in the papers by Caines, Huang and Malham [88], under a smallness assumption on the coefficients or on the time interval) or by Schauder arguments (as in Theorem 4.1, due to Lasry and Lions [104, 103]). Carmona and Delarue [44] use a stochastic maximum principle to derive an MFG system which takes the form of a system of forward-backward stochastic differential equations of a McKean-Vlasov type.

Uniqueness: Concerning the uniqueness of the solution, one can distinguish two kinds of regimes. Of course the Banach fixed point argument provides directly uniqueness of the solution of the MFG system. However, as explained above, it mostly concerns local in time results. For the large time uniqueness, one can rely on the monotonicity conditions (4.12) and (4.13). These conditions first appear in Lasry and Lions [104, 103].

Nash equilibria for the $N-$player games: the use of the MFG system to obtain *$\varepsilon-$Nash equilibria* (Theorem 4.4) has been initiated—in a slightly different framework—in a series of papers due to Caines, Huang and Malham: see in particular [86] (for linear dynamics) and [88] (for nonlinear dynamics). In these papers, the dependence with respect of the empirical measure of dynamics and payoff occurs through an average, so that the CTL implies that the error term is a order $N^{-1/2}$ (instead of $N^{-1/(d+4)}$ as in Theorem 4.4). The genuinely non linear version of the result given above is a variation on a result by Carmon and Delarue [44].

We discuss below the reverse statement: in what extend the MFG system pops up as the limit of Nash equilibria.

Extensions: it is difficult to discuss all the extensions of the MFG systems since the number of papers on this subject has grown exponentially in the last years. We give here only a brief overview.

The ergodic MFG system has been introduced by Lasry and Lions in [105] as the limit, when the number of players tends to infinity, of Nash equilibria in ergodic differential games. As explained in Lions [108], this system also pops up as the limit, as the horizon tends to infinity, of the finite horizon MFG system. We discuss this convergence in the next section, in a slightly simpler setting.

The natural issue of boundary conditions has not been thoroughly investigated up to now. For the PDE approach, the authors have mostly worked with periodic data (as we did above), which completely eliminates this question. In the "probabilistic literature" (as in the work by Caines, Huang and Malham), the natural set-up is the full space. Beside these two extreme cases, little has been written (see however Cirant [48], for Neumann boundary condition in ergodic multi-population MFG systems).

The interesting MFG systems with several populations were introduced in the early paper by Caines, Huang and Malham [88] and revisited by Cirant [48] (for Neuman boundary conditions) and by Kolokoltsov, Li and Yang [95] (for very general diffusions, possibly with jumps).

A very general MFG model for a single population is described in Gomes, Patrizi and

Voskanyan [67] and Gomes and Voskanyan[68], in which the velocity of the population is a nonlocal function of the (repartition of) actions of the players.

5 Second order MFG systems with local coupling

In this part, we concentrate on the MFG system with a local coupling:

$$\begin{array}{ll} (i)\ \ -\partial_t u - \nu\Delta u + H(x,Du) = f(x,m(x,t)) & \text{in } \mathbb{T}^d\times(0,T) \\ (ii)\ \ \partial_t m - \nu\Delta m - \mathrm{div}\,(D_pH(x,Du)m) = 0 & \text{in } \mathbb{T}^d\times(0,T) \\ (iii)\ \ m(0,x)=m_0(x)\ ,\ u(x,T)=G(x). & \end{array} \tag{5.1}$$

Here, the Hamiltonian $H:\mathbb{T}^d\times\mathbb{R}^d\to\mathbb{R}$ is as before but the map $f:\mathbb{T}^d\times[0,+\infty)\to\mathbb{R}$ is now a local coupling between the value function of the optimal control problem and the density of the distribution of the players. We will usually set $\nu=1$. Our aim is first to show that the problem has a unique solution under suitable assumptions on H and a monotonicity condition on f. Then we explain that the system (5.1) can be interpreted as an optimality condition of two optimal control problems of partial differential equations. We complete the section by the analysis of the long time average of the system and its link with the ergodic MFG system.

5.1 Existence of a solution

Let us assume that the coupling $f:\mathbb{T}^d\times[0,+\infty)\to\mathbb{R}$ is smooth (say, C^3) and that the initial and terminal conditions m_0 and G are $C^{2+\beta}$.

Theorem 5.1. *Under the above assumptions, if*

- *either the Hamiltonian is quadratic:* $H(x,p)=\dfrac{1}{2}|p|^2$, *and the coupling* f *is bounded,*
- *or* H *is of the class* C^2 *and globally Lipschitz continuous,*

then (5.1) has at least one classical solution.

Remark 5.2. As we will see later, uniqueness holds if f is strictly increasing in m. Existence of a solution actually holds for quadratic Hamiltonians without a condition on the growth of f, provided that f is bounded below. The proof is, however, much more involved than the one presented here (see [40]).

Proof. For simplicity we set $\nu=1$. We first assume that the Hamiltonian is quadratic and f is bounded. Let us fix a smooth nonnegative kernel $\xi:\mathbb{R}\to\mathbb{R}$ with compact support such that

$$\int_{-\infty}^{\infty} f(\xi)d\xi = 1,$$

and let us set, $\xi_\varepsilon(s)=\varepsilon^{-1}\xi(s/\varepsilon)$. We define, for any $m\in P(\mathbb{T}^d)$,

$$f^\varepsilon(x,m) = f(x,\xi^\varepsilon \star m)$$

As f^ε is regularizing, Theorem 4.1 states that the system

$$\left\{\begin{array}{lll} (i) & -\partial_t u^\varepsilon - \Delta u^\varepsilon + \frac{1}{2}|Du^\varepsilon|^2 = f^\varepsilon(x,m^\varepsilon) & \text{in } \mathbb{T}^d\times(0,T) \\ (ii) & \partial_t m^\varepsilon - \Delta m^\varepsilon - \operatorname{div}(m^\varepsilon Du^\varepsilon) = 0 & \text{in } \mathbb{T}^d\times(0,T) \\ (iii) & m^\varepsilon(0)=m_0\ ,\ u^\varepsilon(x,T)=G(x) & \end{array}\right. \tag{5.2}$$

has at least one classical solution. In order to proceed, one needs estimates on this solution. First note that, in view of the boundedness condition on f, the term $f^\varepsilon(x,m^\varepsilon)$ is uniformly bounded. So, by the maximum principle, the (u^ε) are also uniformly bounded:

$$\|u^\varepsilon\|_\infty \leq C$$

(where C depend on $\|f\|_\infty$ and T). We now use the Hopf-Cole transform, which consists in setting $w^\varepsilon = e^{-u^\varepsilon/2}$. A straightforward computation shows that w^ε solves

$$\left\{\begin{array}{lll} (i) & -\partial_t w^\varepsilon - \Delta w^\varepsilon + w^\varepsilon f^\varepsilon(x,m^\varepsilon) = 0 & \text{in } \mathbb{T}^d\times(0,T) \\ (ii) & w^\varepsilon(x,T) = e^{-G(x)/2} & \end{array}\right. \tag{5.3}$$

Since the (u^ε) are uniformly bounded, so are the (w^ε). Then standard estimates on linear equations (recalled in Theorem **??**—take $a_{ij}=\delta_{ij}$, $a_i=b_i=f_i=f=0$, $a=f^\varepsilon(x,m^\varepsilon)$) imply Hlder bounds on w^ε and Dw^ε:

$$\|w^\varepsilon\|_{C^{\alpha,\alpha/2}} + \|Dw^\varepsilon\|_{C^{\alpha,\alpha/2}} \leq C,$$

where α and C depends only on the bound on f and on the $C^{2+\beta}$ regularity of G. As u^ε is bounded, we immediately derive similar estimates for u^ε:

$$\|u^\varepsilon\|_{C^{\alpha,\alpha/2}} + \|Du^\varepsilon\|_{C^{\alpha,\alpha/2}} \leq C.$$

Next we estimate m^ε: as m^ε solves the linear equation (5.2)-(ii), standard estimate (recalled in Theorem **??**—take $a_{ij}=\delta_{ij}$, $a_i=u^\varepsilon_{x_i}$, $b_i=f_i=f=a=0$) imply that the (m^ε) are bounded in Hlder norm:

$$\|m^\varepsilon\|_{C^{\alpha,\alpha/2}} \leq C.$$

Accordingly the coefficients of (5.3) are bounded in $C^{\alpha,\alpha/2}$. Then Theorem **??** provides $C^{2+\alpha,1+\alpha/2}$ estimate of the solution w^ε, which can be rewritten as an estimate on u^ε:

$$\|u^\varepsilon\|_{C^{2+\alpha,1+\alpha/2}} \leq C.$$

In turn the m^ε solve an equation with Hlder continuous coefficients, therefore one has $C^{2+\alpha,1+\alpha/2}$ estimates on m^ε. So we can extract a subsequence of the $(m^\varepsilon,u^\varepsilon)$ which converges in $C^{2,1}$ to (m,u), where (m,u) is a solution to (5.1).

Let us now explain the variant in which H is of class C^2 and is globally Lipschitz continuous. The technique of proof is basically the same: let $(m^\varepsilon,u^\varepsilon)$ be a solution of the equation with regularizing right-hand side:

$$\left\{\begin{array}{lll} (i) & -\partial_t u^\varepsilon - \Delta u^\varepsilon + H(x,Du^\varepsilon) = f^\varepsilon(x,m^\varepsilon) & \text{in } \mathbb{T}^d\times(0,T) \\ (ii) & \partial_t m^\varepsilon - \Delta m^\varepsilon - \operatorname{div}(m^\varepsilon D_pH(x,Du^\varepsilon)) = 0 & \text{in } \mathbb{T}^d\times(0,T) \\ (iii) & m^\varepsilon(0)=m_0\ ,\ u^\varepsilon(x,T)=G(x) & \end{array}\right. \tag{5.4}$$

As $D_pH(x,Du^\varepsilon)$ is globally bounded, m^ε solves a linear equation with bounded coefficients: therefore m^ε is bounded in Hlder norm. Then we come back to (5.4)-(i), which has a right-hand side bounded in Hlder norm: this implies from Theorem **??** that the solution u^ε is bounded in $C^{2+\alpha,1+\alpha/2}$. One can then conclude as before. □

5.2 Uniqueness of a solution

We now discuss uniqueness issues. For doing so, we work in a very general framework and exhibit a structure condition on a coupled Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$ for uniqueness of classical solutions $(u,m) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^2$ to the local MFG system:

$$\left\{\begin{array}{ll} (i) & -\partial_t u - \nu\Delta u + H(x,Du,m) = 0 \qquad \text{in } \mathbb{T}^d \times (0,T) \\ (ii) & \partial_t m - \nu\Delta m - \operatorname{div}(m\ D_pH(x,Du,m)) = 0 \qquad \text{in } \mathbb{T}^d \times (0,T) \\ (iii) & m(0) = m_0,\ u(x,T) = g(x) \qquad \text{in } \mathbb{T}^d \end{array}\right. \tag{5.5}$$

In the above system, ν is positive, $H = H(x,p,m)$ is a convex Hamiltonian (in p) depending on the density m, $g : \mathbb{T}^d \to \mathbb{T}$ is smooth, m_0 is a probability density on $\mathbb{R}^d$.

Theorem 5.3. *Assume that $H = H(x,p,m)$ is a C^2 function, such that*

$$\begin{pmatrix} m\ \partial^2_{pp}H & \frac{1}{2}m\ \partial^2_{pm}H \\ \frac{1}{2}m\ (\partial^2_{pm}H)^T & -\partial_m H \end{pmatrix} >\ 0 \qquad \forall (x,p,m) \text{ with } m > 0 \tag{5.6}$$

Then system (5.5) has at most one classical solution.

Remark 5.4. 1. Condition (5.6) implies that $H = H(x,p,m)$ is uniformly convex with respect to p and strictly decreasing with respect to m.

2. When H is separate: $H(x,p,m) = \tilde H(x,p) - f(x,m)$, condition (5.6) reduces to $D^2_{pp}\tilde H > 0$ and $D_m f > 0$.

Example 5.5. Assume that H is of the form: $H(x,p,m) = \dfrac{1}{2}\dfrac{|p|^2}{m^\alpha}$, where $\alpha > 0$. Then condition (5.6) holds if and only if $\alpha \in (0,2)$. Note however that existence of solutions is not clear for H of the above form.

Proof. Note that $H = \frac{1}{2}\frac{|p|^2}{m^\alpha}$ is convex in p, decreasing in m if $\alpha > 0$. Moreover

$$\begin{aligned} \left(-\frac{\partial_m H}{m}\right)\ \partial^2_{pp}H - \frac{1}{4}\ \partial^2_{pm}H \otimes \partial^2_{pm}H \ &=\ \frac{\alpha|p|^2}{2m^{\alpha+2}}\frac{I_d}{m^\alpha} - \frac{\alpha^2}{4}\frac{p\otimes p}{m^{2\alpha+2}} \\ &=\ \frac{\alpha|p|^2 I_d}{4m^{2\alpha+2}} - \frac{\alpha^2}{4}\frac{p\otimes p}{m^{2\alpha+2}} \end{aligned}$$

which is positive if and only if $\alpha \in (0,2)$. □

Before starting the proof of Theorem 5.3, let us reformulate condition (5.6) in a more convenient way (omitting the x dependence for simplicity):

Lemma 5.1. *Condition (5.6) implies the inequality*

$$
\begin{aligned}
&(H(p_2, m_2) - H(p_1, m_1))(m_2 - m_1) \\
&\qquad -\langle p_2 - p_1, m_2 D_p H(p_2, m_2) - m_1 D_p H(p_1, m_1)\rangle \geq 0 \ ,
\end{aligned}
\tag{5.7}
$$

with equality if and only if $(m_1, p_1) = (m_2, p_2)$.

Remark 5.6. In fact the above implication is almost an equivalence, in the sense that, if (5.7) holds, then

$$
\begin{pmatrix} m\ \partial^2_{pp} H & \frac{1}{2} m\ \partial^2_{pm} H \\ \frac{1}{2} m\ (\partial^2_{pm} H)^T & -\partial_m H \end{pmatrix} \ \geq \ 0
$$

Proof of Lemma 5.1. Set $\tilde{p} = p_2 - p_1$, $\tilde{m} = m_2 - m_1$ and, for $\theta \in [0,1]$, $p_\theta = p_1 + \theta(p_2 - p_1)$, $m_\theta = m_1 + \theta(m_2 - m_1)$. Let

$$
I(\theta) = (H(p_\theta, m_\theta) - H(x, p_1, m_1))\tilde{m} - \langle \tilde{p}, m_\theta D_p H(Du_\theta, m_\theta) - m_1 D_p H(Du_1, m_1)\rangle
$$

Then $I(0) = 0$ and

$$
I'(\theta) = -\left(\tilde{p}^T\ \tilde{m}\right) \begin{pmatrix} m_\theta\ \partial^2_{pp} H & \frac{1}{2} m_\theta\ \partial^2_{pm} H \\ \frac{1}{2} m_\theta\ (\partial^2_{pm} H)^T & -\partial_m H \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{m} \end{pmatrix}
$$

Hence if condition (5.6) holds and $(p_1, m_1) \neq (p_2, m_2)$, then

$$
0 < I(1) = (H(p_2, m_2) - H(p_1, m_1))(m_2 - m_1) - \langle p_2 - p_1, m_2 D_p H(p_2, m_2) - m_1 D_p H(p_1, m_1)\rangle.
$$

$\square$

Proof of Theorem 5.3. Let (u_1, m_1) and (u_2, m_2) be solutions to (5.5). Let us set

$$
\tilde{m} = m_2 - m_1,\ \tilde{u} = u_2 - u_1,\ \tilde{H} = H(x, Du_2, m_2) - H(x, Du_1, m_1),
$$

$$
\tilde{\mathrm{div}} = \mathrm{div}(m_2 D_p H(x, Du_2, m_2)) - \mathrm{div}(m_1 D_p H(x, Du_1, m_1))
$$

Then

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^d} (u_2(t) - u_1(t))(m_2(t) - m_1(t)) \\
&\qquad = \int_{\mathbb{T}^d} (\partial_t u_2 - \partial_t u_1)(m_2 - m_1) + (u_2 - u_1)(\partial_t m_2 - \partial_t m_1) \\
&\qquad = \int_{\mathbb{T}^d} (-\nu\Delta\tilde{u} + \tilde{H})\tilde{m} + \tilde{u}(\nu\Delta\tilde{m} + \tilde{\mathrm{div}}) \\
&\qquad = \int_{\mathbb{T}^d} \tilde{H}\ \tilde{m} - \langle D\tilde{u}, m_2 D_p H(Du_2, m_2) - m_1 D_p H(Du_1, m_1)\rangle\ \leq\ 0
\end{aligned}
$$

by condition (5.6) and Lemma 5.1. Since $u_1(\cdot,T) = u_2(\cdot,T) = G(\cdot)$ and $m_1(\cdot,0) = m_2(\cdot,0) = m_0(\cdot)$, we have

$$0 \;=\; \left[\int_{\mathbb{T}^d} (u_2(t) - u_1(t))(m_2(t) - m_1(t))\right]_0^T .$$

Integrating $\displaystyle\frac{d}{dt}\int_{\mathbb{T}^d} (u_2(t) - u_1(t))(m_2(t) - m_1(t))$ between 0 and T gives

$$\int_0^T \int_{\mathbb{T}^d} \tilde{H}\ \tilde{m} - \langle D\tilde{u}, m_2 D_p H(Du_2, m_2) - m_1 D_p H(Du_1, m_1)\rangle \;=\; 0.$$

In view of Lemma 5.1 again, this implies that $D\tilde{u} = 0$ and $\tilde{m} = 0$, so that $m_1 = m_2$ and $u_1 = u_2$. □

5.3 Optimal control interpretation

Here we show that the MFG system (5.1) can be viewed as an optimality condition for two optimal control problems: the first one is an optimal control of Hamilton-Jacobi equations and the second one concerns the optimal control of the Fokker-Planck equation.

In order to do so, let us first introduce some assumptions and notations: without loss of generality, we suppose that $f(x,0) = 0$: indeed we can always subtract $f(x,0)$ to both sides of (5.1). Let

$$F(x,m) = \begin{cases} \displaystyle\int_0^m f(x,\rho)d\rho & \text{if } m \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

As f is nondecreasing with respect to the second variable, $F = F(x,m)$ is convex with respect to m. We denote by $F^* = F^*(x,\alpha)$ its convex conjugate:

$$F^*(x,\alpha) = \sup_{m\in\mathbb{R}} \left(\alpha m - F(x,m)\right) \qquad \forall (x,\alpha) \in \mathbb{T}^d \times \mathbb{R}\ .$$

Note that F^* is convex and nondecreasing with respect to the second variable. We also introduce the convex conjugate $H^*(x,\xi)$ of the map $H = H(x,p)$ with respect to the second variable:

$$H^*(x,\xi) = \sup_{p\in\mathbb{R}^d} \left(\langle \xi, p\rangle - H(x,p)\right) \qquad \forall (x,\xi) \in \mathbb{T}^d \times \mathbb{R}^d\ .$$

We assume throughout this section that F^* and H^* are smooth enough to perform the computations.

The first optimal control we consider is the following: the distributed control parameter is $\alpha : \mathbb{T}^d \times [0,T] \to \mathbb{R}$ and the state parameter is u. We aim at minimizing the criterium

$$\mathcal{J}^{HJ}(\alpha) = \int_0^T \int_{\mathbb{T}^d} F^*\left(x, \alpha(x,t)\right)\ dxdt - \int_{\mathbb{T}^d} u(0,x) dm_0(x).$$

over Lipschitz continuous maps $\alpha : \mathbb{T}^d \times (0,T) \to \mathbb{R}^d$, where u is the unique classical solution to the backward Hamilton-Jacobi equation

$$\begin{cases} -\partial_t u(x,t) - \Delta u(x,t) + H(x, Du(x,t)) = \alpha(x,t) & \text{in } \mathbb{T}^d \times (0,T) \\ u(x,T) = G(x) \quad \text{in } \mathbb{T}^d \end{cases} \tag{5.8}$$

The second optimal control problem is related with Fokker-Planck equation: the (distributed and vector valued) control is now $v : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ and the state is m. It consists in minimizing the criterium

$$\mathcal{J}^{FP}(v) = \int_0^T \int_{\mathbb{T}^d} m(x,t) H^*\left(x, -v(x,t)\right) + F(x, m(x,t))\, dxdt + \int_{\mathbb{T}^d} G(x) m(T,x) dx,$$

where the pair (m, v) solves the Fokker-Planck equation

$$\partial_t m - \Delta m(x,t) + \operatorname{div}(mv) = 0 \text{ in } \mathbb{T}^d \times (0,T), \qquad m(0) = m_0. \tag{5.9}$$

Theorem 5.7. *Assume that $(\bar{m}, \bar{u})$ is of class $C^2(\mathbb{T}^d \times [0,T])$, with $\bar{m}(x,0) = m_0$ and $\bar{u}(x,T) = G(x)$. Suppose furthermore that $\bar{m}(x,t) > 0$ for any $(x,t) \in \mathbb{T}^d \times [0,T]$. Then the following statements are equivalent:*

(i) $(\bar{u}, \bar{m})$ is a solution of the MFG system (5.1).

(ii) The control $\bar{\alpha}(x,t) := f(x, \bar{m}(x,t))$ is optimal for $\mathcal{J}^{HJ}$ and the solution to (5.8) is given by $\bar{u}$.

(iii) The control $\bar{v}(x,t) := -D_pH(x, D\bar{u}(x,t))$ is optimal for $\mathcal{J}^{FP}$, $\bar{m}$ being the solution of (5.9).

Remark 5.8.

1. The optimal control problem of Hamilton-Jacobi equation can be rewritten as

$$\inf_u \int_0^T \int_{\mathbb{T}^d} F^*\left(x, -\partial_t u(x,t) - \Delta u(x,t) + H(x, Du(x,t))\right)\, dxdt - \int_{\mathbb{T}^d} u(0,x) dm_0(x)$$

under the constraint that u sufficiently smooth, with $u(\cdot, T) = G(\cdot)$. Remembering that H is convex with respect to the last variable and that F is convex and increasing with respect to the last variable, it is clear that the above problem is convex.

2. The optimal control problem of the Fokker-Planck equation is also a convex problem, up to a change of variables which appears frequently in optimal transportation theory: let us set $w = mv$. Then the problem can be rewritten as

$$\inf_{(m,w)} \int_0^T \int_{\mathbb{T}^d} m(x,t) H^*\left(x, -\frac{w(x,t)}{m(x,t)}\right) + F(x, m(x,t))\, dxdt + \int_{\mathbb{T}^d} G(x) m(T,x) dx,$$

where the pair (m, w) solves the Fokker-Planck equation

$$\partial_t m - \Delta m(x,t) + \operatorname{div}(w) = 0 \text{ in } \mathbb{T}^d \times (0,T), \qquad m(0) = m_0. \tag{5.10}$$

This problem is convex because the constraint (5.10) is linear and the map $(m, w) \to mH^*\left(x, -\frac{w}{m}\right)$ is convex on $\mathbb{T}^d \times (0, +\infty)$.

3. In fact the two optimal control problems just defined are conjugate in the Fenchel-Rockafellar sense (see, for instance, [52] Ekeland).

Proof. The proof is done by verification. We will show only the equivalence between (i) and (ii): the equivalence between (i) and (iii) can be established in a symmetrical way, by using the reformulation given in Remark (5.8). Let us first assume that $(\bar{m}, \bar{u})$ is a solution of (5.1). Let α be a Lipschitz continuous map and u the corresponding solution of (5.8). Then, by (5.8),

$$\begin{aligned}
\mathcal{J}^{HJ}(\alpha) = & \int_0^T \int_{\mathbb{T}^d} F^*\left(x, -\partial_t u(x,t) - \Delta u(x,t) + H(x, Du(x,t))\right)\, dxdt - \int_{\mathbb{T}^d} u(0,x) dm_0(x) \\
\geq & \ \mathcal{J}^{HJ}(\bar{\alpha}) + \int_0^T \int_{\mathbb{T}^d} \partial_\alpha F^*(x, \bar{\alpha}) \left(-\partial_t (u - \bar{u}) - \Delta(u - \bar{u}) + \langle D_p H(x, D\bar{u})), D(u - \bar{u}) \rangle\right) \\
& - \int_{\mathbb{T}^d} (u - \bar{u})(0,x) dm_0(x)
\end{aligned}$$

where, to get the inequality, we have used the convexity of the map

$$u \to F^*\left(x, -\partial_t u(x,t) - \Delta u(x,t) + H(x, Du(x,t))\right)$$

which holds because F^* is convex and nondecreasing with respect to the second variable. Now we note that, by equality $\bar{\alpha}(x,t) := f(x, \bar{m}(x,t))$ and property of Legendre transform, we actually have $\partial_\alpha F^*(x, \bar{\alpha}(x,t)) = \bar{m}(x,t)$. So

$$\begin{aligned}
\mathcal{J}^{HJ}(\alpha) \geq & \ \mathcal{J}^{HJ}(\bar{\alpha}) + \int_0^T \int_{\mathbb{T}^d} \bar{m} \left(-\partial_t (u - \bar{u}) - \Delta(u - \bar{u}) + \langle D_p H(x, D\bar{u})), D(u - \bar{u}) \rangle\right) \\
& - \int_{\mathbb{T}^d} (u - \bar{u})(0,x) dm_0(x)
\end{aligned}$$

Integrating by parts we get

$$\begin{aligned}
\mathcal{J}^{HJ}(\alpha) \geq & \ \mathcal{J}^{HJ}(\bar{\alpha}) + \int_0^T \int_{\mathbb{T}^d} (u - \bar{u}) \left(\partial_t \bar{m} - \Delta \bar{m} - \operatorname{div}(\bar{m} D_p H(x, D\bar{u}))\right) \\
& + \int_{\mathbb{T}^d} (u - \bar{u})(T,x) m(x,T) dx \\
\geq & \ \mathcal{J}^{HJ}(\bar{\alpha})
\end{aligned}$$

where the last inequality comes from the equation satisfied by $\bar{m}$ and the fact that $u(x,T) = \bar{u}(x,T) = G(x)$. So we have proved that $\bar{\alpha}$ is optimal for $\mathcal{J}^{HJ}$.

Conversely, let us assume that the control $\bar{\alpha}$ is optimal in $\mathcal{J}^{HJ}$. Let us set $\bar{m}(x,t) = \partial_\alpha F^*(x, \bar{\alpha}(x,t))$, i.e., $\bar{\alpha}(x,t) := f(x, \bar{m}(x,t))$. We want to show that the pair $(\bar{u}, \bar{m})$ is a solution to (5.1). For this, let $a \in C^1(\mathbb{T}^d \times [0,T])$ and, for $h \neq 0$, let u_h be the solution of (5.8) associated to the control $\bar{\alpha} + ha$. Then $(u_h - \bar{u})/h$ converge to some w in $C^{2,1}$, where w solves the linearized system

$$\begin{cases} -\partial_t w(x,t) - \Delta w(x,t) + \langle D_p H(x, D\bar{u}(x,t)), w(x,t) \rangle = a(x,t) & \text{in } \mathbb{T}^d \times (0,T) \\ w(x,T) = 0 \quad \text{in } \mathbb{T}^d \end{cases} \tag{5.11}$$

Using the optimality of $\bar\alpha$, we obtain

$$0 = d\mathcal{J}^{HJ}(\bar\alpha)(a) = \int_0^T\int_{\mathbb{T}^d} \bar m \left(-\partial_t w - \Delta w + \langle D_pH(x,D\bar u)), Dw\rangle\right) - \int_{\mathbb{T}^d} w(0,x)dm_0(x).$$

We integrate by parts to get, as $w(x,T)=0$,

$$0 = \int_0^T\int_{\mathbb{T}^d} w\left(\partial_t \bar m - \Delta \bar m - \mathrm{div}(\bar m D_pH(x,D\bar u))\right) - \int_{\mathbb{T}^d} w(0,x)(m_0(x)-m(x,0))dx. \quad (5.12)$$

Note that if ones fixes $w\in C^3$ such that $w(x,T)=0$, we can always define a in such a way that (5.11) holds. By density, this implies that relation (5.12) also holds for any $w\in C^3$ such that $w(x,T)=0$ and therefore that $\bar m$ is a weak solution of (5.1)-(ii) with $\bar m(x,0)=m_0(x)$. □

5.4 The long time average

In this section we study the long time average of solutions of the MFG system (5.1). We concentrate on the simple case $H(x,p)=\frac{1}{2}|p|^2$. We also suppose that the coupling $f=f(x,m)$ is bounded and strictly increasing with respect to the last variable:

$$\frac{\partial f}{\partial m}(x,m) > 0. \quad (5.13)$$

As in the previous section, we suppose without loss of generality that f is non negative. Moreover, we assume that the initial density m_0 is positive and smooth.

To emphasize the fact that we are interested in the behavior of the solution as the horizon T tends to $+\infty$, we denote by (u^T,m^T) the solution to

$$\left\{\begin{array}{rl} (i) & -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = f(x,m(x,t)) \qquad \text{in } \mathbb{T}^d\times(0,T)\\ (ii) & \partial_t m - \Delta m - \mathrm{div}\,(mDu) = 0 \qquad \text{in } \mathbb{T}^d\times(0,T)\\ (iii) & m(0)=m_0\ ,\ u(x,T)=G(x)\end{array}\right. \quad (5.14)$$

It is expected that the MFG ergodic system should play a key role in this problem. The ergodic system, with unknowns $(\bar\lambda,\bar u,\bar m)$, is

$$\left\{\begin{array}{rl} (i) & \bar\lambda - \Delta \bar u + \frac{1}{2}|D\bar u|^2 = f(x,\bar m)\\ (ii) & -\Delta\bar m - \mathrm{div}(\bar m D\bar u) = 0\\ (iii) & \int_{\mathbb{T}^d}\bar u\,dx = 0\,,\quad \int_{\mathbb{T}^d}\bar m\,dx = 1\end{array}\right. \quad (5.15)$$

Let us first remark that the above system is well-defined:

Proposition 5.9. *Under the assumptions of this section, system (5.15) has a unique classical solution $(\bar\lambda,\bar u,\bar m)$. Moreover $\bar m = e^{-\bar u}/\left(\int_{\mathbb{T}^d} e^{-\bar u}\right) > 0$.*

The proof can be established by usual fixed point arguments so we omit it.

In order to understand to what extent the solution $(\bar{\lambda}, \bar{u}, \bar{m})$ of (5.15) drives the behavior of (u^T, m^T), let us introduce the scaled functions

$$\upsilon^T(x,s) := u^T(x, sT) \qquad ; \qquad \mu^T(x,s) := m^T(x,sT) \qquad (x,s) \in \mathbb{T}^d \times [0,1] \,. \tag{5.16}$$

Theorem 5.10. *As $T \to +\infty$, the map $(x,s) \to \upsilon^T(x,s)/T$ converges to the (space independent) map $(x,s) \to (1-s)\bar{\lambda}$ in $L^2(\mathbb{T}^d \times (0,1))$.*

Remark 5.11. With more estimates than presented here, one can show that the map μ^T converges to $\bar{m}$ in $L^p(\mathbb{T}^d \times (0,1))$, for any $p < \frac{d+2}{d}$.

The proof of Theorem 5.10 requires several intermediate steps. The starting point is the usual estimate, which is crucial in establishing the uniqueness of the solution to (5.14).

Lemma 5.2. *For any $0 \le t_1 < t_2 \le T$ we have*

$$\begin{aligned}&\left[\int_{\mathbb{T}^d} (u^T - \bar{u})(m^T - \bar{m})dx\right]_{t_1}^{t_2} \\ &+ \int_{t_1}^{t_2}\int_{\mathbb{T}^d} \frac{(m^T + \bar{m})}{2}|Du^T - D\bar{u}|^2 + (f(x,m^T) - f(x,\bar{m}))(m^T - \bar{m})\ dxdt = 0\end{aligned}$$

Proof. Since T is fixed, we simply write m and u instead of m^T and u^T. We first integrate over $\mathbb{T}^d \times (t_1, t_2)$ the equation satisfied by $(u - \bar{u})$ multiplied by $(m - \bar{m})$. Since $\int_{\mathbb{T}^d}(m(t,x) - \bar{m}(x))dx = 0$ and $\bar{u}$ does not depend on time, we get, after integration by parts:

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{T}^d} -\partial_t u(m - \bar{m}) + \langle D(m - \bar{m}), D(u - \bar{u})\rangle + \frac{1}{2}(m - \bar{m})(|Du|^2 - |D\bar{u}|^2) \\ = \int_{t_1}^{t_2}\int_{\mathbb{T}^d} (f(x,m) - f(x,\bar{m}))(m - \bar{m}) \,.\end{aligned}$$

In the same way we integrate over $\mathbb{T}^d \times (t_1, t_2)$ the equation satisfied by $(m - \bar{m})$ multiplied by $(u - \bar{u})$:

$$\int_{t_1}^{t_2}\int_{\mathbb{T}^d} (u - \bar{u})\partial_t m + \langle D(m - \bar{m}), D(u - \bar{u})\rangle + \langle mDu - \bar{m}D\bar{u}, D(u - \bar{u})\rangle = 0 \,.$$

We now compute the difference between the second equation and the first one:

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{T}^d} \partial_t[(u - \bar{u})(m - \bar{m})] + \langle mDu - \bar{m}D\bar{u}, D(u - \bar{u})\rangle \\ -\frac{1}{2}(m - \bar{m})(|Du|^2 - |D\bar{u}|^2) + (f(x,m) - f(x,\bar{m}))(m - \bar{m}) = 0 \,.\end{aligned}$$

To complete the proof we just note that

$$\langle mDu - \bar{m}D\bar{u}, D(u - \bar{u})\rangle - \frac{1}{2}(m - \bar{m})(|Du|^2 - |D\bar{u}|^2) = \frac{(m + \bar{m})}{2}|Du - D\bar{u}|^2 \,.$$

□

Another crucial point is given by the following lemma, which exploits the fact that system (5.14) has an Hamiltonian structure. Note that this is directly related to the optimal control interpretation of the MFG as explained in the previous section.

Lemma 5.3. *There exists a constant M^T such that*

$$\frac{1}{2}\int_{\mathbb{T}^d} m^T(t)\,|Du^T(t)|^2\,dx \qquad + \int_{\mathbb{T}^d}\langle Du^T(t), Dm^T(t)\rangle\,dx$$
$$- \int_{\mathbb{T}^d} F(x, m^T(t))\,dx = M^T \qquad \forall t \in [0,T]$$

where $F(x,m) = \int_0^m f(x,\rho)\,d\rho$.

Proof. We multiply (5.14)-(i) by $\partial_t m^T(t)$ and (5.14)-(ii) by $\partial_t u^T(t)$. Summing the two equations we get, at (t,x),

$$-\Delta u^T \partial_t m^T \ + \frac{1}{2}|Du^T|^2\,\partial_t m^T - f(x,m^T)\partial_t m^T = \Delta m^T \partial_t u^T + \mathrm{div}(m^T Du^T)\partial_t u^T$$

Integrating with respect to x gives:

$$\int_{\mathbb{T}^d} \qquad \big(\langle Du^T, \partial_t Dm^T\rangle + \langle Dm^T, \partial_t Du^T\rangle\big)\,dx$$
$$+ \int_{\mathbb{T}^d}\left[\tfrac{1}{2}|Du^T|^2\,\partial_t m^T + m^T\,\langle Du^T, \partial_t Du^T\rangle\right]\,dx - \int_{\mathbb{T}^d} f(x,m^T)\partial_t m^T\,dx = 0$$

This means that

$$\frac{d}{dt}\left\{\int_{\mathbb{T}^d}\langle Du^T, Dm^T\rangle\,dx + \frac{1}{2}\int_{\mathbb{T}^d} m^T|Du^T|^2\,dx - \int_{\mathbb{T}^d} F(x,m^T)\,dx\right\} = 0\,,$$

hence the conclusion. □

We deduce the following

Corollary 5.4. *We have*

(i) M^T is bounded with respect to T.

(ii) $|Du^T(0)|$ is bounded in $L^2(\mathbb{T}^d)$.

Proof. On one hand we have (since $f \geq 0$ and $u(T) = G(x)$)

$$\begin{aligned} M^T &= \int_{\mathbb{T}^d}\langle Du(T), Dm(T)\rangle\,dx + \frac{1}{2}\int_{\mathbb{T}^d} m(T)|Du(T)|^2\,dx - \int_{\mathbb{T}^d} F(x,m(T))\,dx \\ &\leq -\int_{\mathbb{T}^d}\Delta u(T)\,m(T)\,dx + \frac{1}{2}\int_{\mathbb{T}^d} m(T)|Du(T)|^2\,dx \\ &\leq (\|\Delta G\|_\infty + \|DG\|_\infty^2)\|m(T)\|_{L^1(\mathbb{T}^d)} = C\,. \end{aligned}$$

On the other hand we have

$$M^T = \int_{\mathbb{T}^d}\langle Du(0), Dm_0\rangle\,dx + \frac{1}{2}\int_{\mathbb{T}^d} m_0|Du(0)|^2\,dx - \int_{\mathbb{T}^d} F(x,m_0)\,dx$$

where, since $m_0 > 0$,

$$\left| \int_{\mathbb{T}^d} \langle Du(0), Dm_0 \rangle dx \right| \leq \frac{1}{4} \int_{\mathbb{T}^d} m_0 |Du(0)|^2 \, dx + \int_{\mathbb{T}^d} \frac{|Dm_0|^2}{m_0} \, dx \, .$$

Hence

$$M^T \geq \frac{1}{4} \int_{\mathbb{T}^d} m_0 |Du(0)|^2 \, dx - C \, .$$

In particular M^T is bounded both from above and from below. We also deduce from our last inequality that

$$\int_{\mathbb{T}^d} |Du(0)|^2 \, dx \leq C \, ,$$

so that $|Du(0)|$ is bounded in $L^2(\mathbb{T}^d)$. □

Combining Corollary 5.4 with Lemma 5.2 we get:

Lemma 5.5.

$$\int_0^T \int_{\mathbb{T}^d} \frac{(m^T + \bar{m})}{2} |Du^T - D\bar{u}|^2 + (f(x, m^T) - f(x, \bar{m}))(m^T - \bar{m}) \; dxdt \leq C \tag{5.17}$$

In particular

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_{\mathbb{T}^d} |Du^T - D\bar{u}|^2 \; dxdt = 0 \, . \tag{5.18}$$

Proof. Using Lemma 5.2, we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^d} \frac{(m^T + \bar{m})}{2} |Du^T - D\bar{u}|^2 + (f(x, m^T) - f(x, \bar{m}))(m^T - \bar{m}) \; dxdt \\ &= \int_{\mathbb{T}^d} (u^T(0) - \bar{u})(m_0 - \bar{m}) dx - \int_{\mathbb{T}^d} (u^T(T) - \bar{u})(m^T(T) - \bar{m}) dx \end{aligned}$$

Recalling that $u^T(T) = G$ and the bounds assumed on G, the last term is bounded. If we set $\tilde{u}^T = \int_{\mathbb{T}^d} u^T \, dx$, we have

$$\begin{aligned} \int_{\mathbb{T}^d} u^T(0)(m_0 - \bar{m}) dx \quad &= \textstyle\int_{\mathbb{T}^d} (u^T(0) - \tilde{u}^T(0))(m_0 - \bar{m}) dx & (5.19) \\ &\leq C \left(\|m_0\|_\infty + \|\bar{m}\|_\infty \right) \|Du^T(0)\|_{L^2(\mathbb{T}^d)} & (5.20) \end{aligned}$$

and using Corollary 5.4 we conclude that this is bounded. Therefore, we obtain that (5.17) holds. □

Rewriting Lemma 5.5 in terms of υ^T and μ^T we obtain:

Corollary 5.6. *The maps $(D\upsilon^T)$ and $(f(\cdot, \mu^T(\cdot, \cdot))$ converge to $D\bar{u}$ and $f(\cdot, \bar{m}(\cdot))$ in $L^2(\mathbb{T}^d \times (0,1))$ and in $L^1(\mathbb{T}^d \times (0,1))$ as $T \to +\infty$.*

Proof. Since $\bar{m}$ is bounded below by a positive constant, Lemma 5.5 implies that

$$\int_0^1 \int_{\mathbb{T}^d} |D\upsilon^T - D\bar{u}|^2 \, dxdt \leq \frac{C}{T}$$

Whence the convergence of $(D\upsilon^T)$. From assumption (5.13), there exists $\delta > 0$ such that

$$\frac{\partial f}{\partial m}(x,m) \geq \delta \qquad \forall (x,m) \in \mathbb{T}^d \times [0, 2\|\bar{m}\|_\infty].$$

So, from (5.17),

$$\begin{aligned} \frac{C}{T} &\geq \int_0^T \int_{\mathbb{T}^d} (f(x,\mu^T) - f(x,\bar{m}))(\mu^T - \bar{m}) \, dxdt \\ &\geq \iint_{\{\mu^T \geq 2\|\bar{m}\|_\infty\}} \left| f(x,\mu^T) - f(x,\bar{m}) \right| \|\bar{m}\|_\infty \, dxdt + \delta \iint_{\{\mu^T < 2\|\bar{m}\|_\infty\}} \left| \mu^T - \bar{m} \right|. \end{aligned}$$

Therefore

$$\begin{aligned} &\|f(\cdot,\mu^T) - f(\cdot,\bar{m})\|_1 \\ &\quad \leq \iint_{\{\mu^T \geq 2\|\bar{m}\|_\infty\}} \left| f(x,\mu^T) - f(x,\bar{m}) \right| + \iint_{\{\mu^T < 2\|\bar{m}\|_\infty\}} \left| f(x,\mu^T) - f(x,\bar{m}) \right| \\ &\quad \leq \iint_{\{\mu^T \geq 2\|\bar{m}\|_\infty\}} \left| f(x,\mu^T) - f(x,\bar{m}) \right| + \sup_{0 \leq m \leq 2\|\bar{m}\|_\infty} \left| \frac{\partial f}{\partial m} \right| \iint_{\{\mu^T < 2\|\bar{m}\|_\infty\}} \left| \mu^T - \bar{m} \right| \\ &\quad \leq \frac{C}{T} \left(\frac{1}{\|\bar{m}\|_\infty} + \frac{1}{\delta} \sup_{0 \leq m \leq 2\|\bar{m}\|_\infty} \left| \frac{\partial f}{\partial m} \right| \right) \end{aligned}$$

which implies the convergence of $f(\cdot,\mu^T)$ to $f(\cdot,\bar{m})$ in L^1. □

Proof of Theorem 5.10. We now prove the convergence of υ^T/T to $\bar{\lambda}(1-s)$. Let us integrate the equation satisfied by υ^T on $\mathbb{T}^d \times (t,1)$:

$$\frac{1}{T}\left(\int_{\mathbb{T}^d} \upsilon^T(x,t)dx - \int_{\mathbb{T}^d} G(x)dx \right) \quad + \tfrac{1}{2} \int_t^1 \int_{\mathbb{T}^d} |D\upsilon^T|^2 dxds \tag{5.21}$$

$$= \int_t^1 \int_{\mathbb{T}^d} f(x,\mu^T(s))dxds \tag{5.22}$$

where, from Corollary 5.6, $D\upsilon^T \to D\bar{u}$ in L^2 and $f(\cdot,\mu^T(\cdot)) \to f(\cdot,\bar{m})$ in L^1. So

$$\lim_{T\to+\infty} \frac{1}{T} \int_{\mathbb{T}^d} \upsilon^T(x,t)dx = (1-t) \int_{\mathbb{T}^d} \left[-\frac{1}{2}|D\bar{u}|^2 + f(x,\bar{m}) \right] dx = (1-t)\bar{\lambda} \,,$$

the last equality being obtained by integrating over $\mathbb{T}^d$ equation (5.15)-(i). Using Poincaré-Wirtinger inequality, we get, setting $\langle \upsilon^T \rangle = \int_{\mathbb{T}^d} \upsilon^T dx$ and $\tilde{\upsilon}^T = \upsilon^T - \langle \upsilon^T \rangle$,

$$\int_0^1 \int_{\mathbb{T}^d} \left| \tilde{\upsilon}^T - \bar{u} \right|^2 \leq C \int_0^1 \int_{\mathbb{T}^d} |D(\upsilon^T - \bar{u})|^2 \to 0.$$

This shows the convergence in L^2 of $\tilde{\upsilon}^T$ to $\bar{u}$ and, since $\frac{\langle \upsilon^T \rangle}{T} \to (1-t)\bar{\lambda}$, the convergence in L^2 of $\frac{1}{T}\upsilon^T$ to $(1-t)\bar{\lambda}$. □

5.5 Comments

Other existence results of classical solutions of second order MFG systems with local coupling can be found in Cardaliaguet, Lasry, Lions and Porretta [40] (for quadratic Hamiltonian, without conditions on the coupling f) and for more general Hamiltonians under various structure conditions on the coupling in a series of papers by Gomes, Pires and Sanchez-Morgado [66, 69, 71] and Gomes and Pimentel [73]. The case of MFG system with congestion is considered in Gomes and Mitake [74].

Even for some data, it is not known if there always exists a classical solution to the MFG system. To overcome this issue, concepts of weak solutions have been introduced in Lasry and Lions [103] and in Porretta [117].

The general uniqueness criterium given in Theorem 5.3 has been introduced by Lions [108], who explains the sharpness of the condition.

The fact that the MFG system with local coupling possesses a variational structure is pointed out in Lasry and Lions in [103]. This plays a key role for the first order MFG system with local coupling, since this allows to build solutions in that setting.

Finally the long time behavior of the MFG system is described in section 5.4 has been first discussed by Lions in [108] and sharpened in Cardaliaguet, Lasry, Lions and Porretta [41]. Other results in that direction can be found in Gomes, Mohr and Suza [63] (for discrete MFG systems), Cardaliaguet, Lasry, Lions and Porretta [40] (for MFG system with a nonlocal coupling), Cardaliaguet [37] (for the first order MFG with a nonlocal coupling) and in Cardaliaguet and Graber [36] (for the first order MFG with local coupling). For second order MFG systems, the rate of this convergence is exponential (see [40, 41]).

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